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Ultradiscretization of the theta function solution of pd Toda

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Abstract

A periodic box-ball system (pBBS) is obtained by ultradiscretizing the periodic discrete Toda equation (pd Toda equation). We show the relation between a Young diagram of the pBBS and a spectral curve of the pd Toda equation. The formula for the fundamental cycle of the pBBS is obtained as a corollary.

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1. Preface

A cellular automaton (CA) is a discrete dynamical system which consists of an array of a number of cells. Each cell allows for finitely many states which change into new states in discrete time. Usually the rule of time evolution with which the system is equipped is quite simple, and CAs are often investigated as simple models for natural or social phenomena. The box-ball system (BBS) is one type of CA, represented by finitely many balls and countably many boxes arranged in a line.

In this paper, we study a periodic box-ball system (pBBS), which is a BBS with a periodic boundary condition. The pBBS is closely related to integrable nonlinear equations. In fact, the pBBS has soliton-like solutions and a large number of conserved quantities. Moreover, the pBBS can be obtained from integrable equations by the method of 'ultra discretization'.

This relation gives us a new method to describe the behaviour of a pBBS. If the initial-value problem of integrable equations related to the pBBS is solvable by some analytical method, the initial-value problem of the pBBS itself is also solvable, as the solution of the pBBS is obtained from the solution of the integrable equations by ultradiscretization.

The present paper is organized as follows. In section 2, we introduce the definition of the pBBS and the pd Toda equation. These two objects are connected with each other through 'ultradiscretization'. We define the conserved quantities of these two systems and state a main theorem (theorem 2.3) which yields direct relation between the spectral curve and the Young

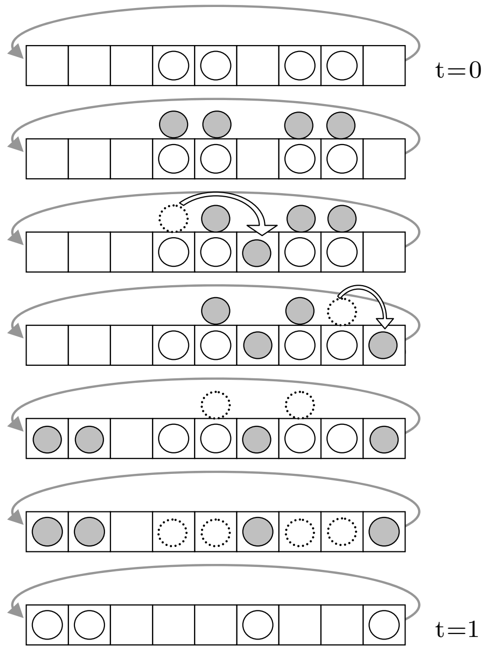

Figure 1. Time evolution rule for pBBS.

diagram. Section 3 proves theorem 2.3. In sections 4 and 5, we give the solution of the initial value problem of the pBBS and derive the fundamental period for it, as a corollary of theorem 2.3.

2. Periodic box-ball system and the periodic discrete Toda equation

2.1. pBBS

Let us consider a one-dimensional array of L boxes. Let Q be the total number of balls, such that $Q < L/2$. Each of these boxes is either empty or is filled with a ball. Since we are interested in the periodic case, the Nth box is adjacent to the first box. The time evolution of this system is

(i) In each filled box, create a copy of the ball.
(ii) Move all copies once according to the following rules.
(iii) Choose one of the copies and move it to the nearest empty box on the right of it.
(iv) Choose one of the remaining copies and move it to the nearest empty box on the right of it.
(v) Repeat the above procedure until all the copies have been moved.
(vi) Delete all the original balls.

It is not difficult to confirm that the resulting state does not depend on the choice of the copies. This dynamical system is called the periodic box-ball system, or pBBS. Figure 1 shows an example of the pBBS and its time evolution pattern. The last entry is considered to be adjacent to the first entry. The pBBS is usually regarded as a dynamical system of a finite

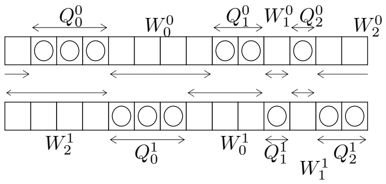

Figure 2. The definition of Q_j^t and W_j^t.

sequence with periodic boundary condition. Let us denote an empty box by '0' and a filled box by '1'.

Let N be the number of groups of consecutive 1's at $t = 0$. (Clearly, N is also the number of groups of consecutive 0's at $t = 0$.) This number N does not change under the time evolution and it corresponds to the number of solitons in the pBBS. We introduce dependent variables $Q_j^t, W_j^t (j = 1, \ldots, N, t \in \mathbb{N})$, as in figure 2.

At $t = 0$, choose one of the sets of consecutive 1's and denote the number of 1's by Q_0^0. Next, looking to the right, denote the number of 0's in the nearest set of consecutive 0's by W_0^0. Then, looking to the right, denote the number of 1's in the nearest set of consecutive 1's by Q_1^0. We continue to define $W_1^0, Q_2^0, \ldots, Q_N^0, W_N^0$ in a similar manner. Since our system has the periodic boundary condition, it follows $Q_N^0 = Q_0^0, W_N^0 = W_0^0, \ldots$ etc. In the following, we always use the convention that the position j is defined in $\mathbb{Z}_N$ (i.e. $Q_{j+N}^t = Q_j^t, W_{j+N}^t = W_j^t$). At $t = 1$, to define $Q_0^1, W_0^1, \ldots$, etc, one needs to choose one of the sets of consecutive 1's; The set of consecutive 1's whose leftmost entry was updated from the 0' of the first set of consecutive 0's will be called Q_0^1. In general, Q_0^{t+1} is defined as the number of entries in the set of consecutive 1's whose leftmost entry was updated from the 0' of the first set of consecutive 0's at t.

The following formulae describe the time evolution of the pBBS:

$$Q_i^{t+1} = \min\left[W_i^t, X_i^t + Q_i^t\right] \tag{2.1}$$

$$W_i^{t+1} = Q_{i+1}^t + W_i^t - Q_i^{t+1} \tag{2.2}$$

$$X_i^t = \max_{k=0,\ldots,N-1}\left[\sum_{l=1}^{k}\left(Q_{i-l}^t - W_{i-l}^t\right)\right], \tag{2.3}$$

where it follows that

$$\sum_{i=1}^{N} Q_i^t < \sum_{i=1}^{N} W_i^t \tag{2.4}$$

due to the condition $Q < L/2$.

The main feature we use to solve the initial value problem of the pBBS is the correspondence between the pBBS and the periodic Toda equation.

Definition 2.1. *The periodic Toda equation (pd Toda equation) is given as*

$$I_i^{t+1} = I_i^t + V_i^t - V_{i-1}^{t+1} \tag{2.5}$$

$$V_i^{t+1} = \frac{I_{i+1}^t V_i^t}{I_i^{t+1}} \tag{2.6}$$

with the boundary condition

$$I^t_{i+N} = I^t_i, \qquad V^t_{i+N} = V^t_i. \tag{2.7}$$

The following proposition shows the essential relation between the pBBS and the pd Toda equation.

Proposition 2.1 ([1]). *Suppose that the pd Toda equation has a one-parameter family of real positive solutions* $\left\{I^t_j(\varepsilon), V^t_j(\varepsilon)\right\}_{\varepsilon>0}$. *If a solution to the pd Toda equation satisfies*

$$0 \leqslant \lim_{\varepsilon\to 0^+} \frac{\prod_{i=1}^N V^t_i}{\prod_{i=1}^N I^t_i} < 1, \tag{2.8}$$

and if the limits

$$Q^t_j \equiv \lim_{\varepsilon\to 0^+} -\varepsilon \log I^t_j(\varepsilon) \qquad \textit{and} \qquad W^t_j \equiv \lim_{\varepsilon\to 0^+} -\varepsilon \log V^t_j(\varepsilon)$$

exist, they satisfy equations (2.1), (2.2), (2.3) and (2.4).

Proof. Substituting (2.6) to (2.5), we have

$$I^{t+1}_i = I^t_i + V^t_i - \frac{I^t_i V^t_{i-1}}{I^{t+1}_{i-1}}.$$

Since I^{t+1}_{i-1} satisfies the same equation,

$$I^{t+1}_i = I^t_i + V^t_i - \frac{I^t_i V^t_{i-1}}{I^t_{i-1} + V^t_{i-1} - \frac{I^t_{i-1} V^t_{i-2}}{I^{t+1}_{i-2}}}.$$

Repeating this procedure, we get the following equation due to the periodic boundary condition:

$$I^{t+1}_i = I^t_i + V^t_i - \cfrac{I^t_i V^t_{i-1}}{I^t_{i-1} + V^t_{i-1} - \cfrac{I^t_{i-1} V^t_{i-2}}{I^t_{i-2} + V^t_{i-2} - \cfrac{I^t_{i-2} V^t_{i-3}}{I^t_{i-3} + V^t_{i-3} - \cfrac{I^t_{i-3} V^t_{i-4}}{\ddots \; \frac{I^t_{i+1} V^t_i}{I^{t+1}_i}}}}}.$$

This is a quadratic equation of I^{t+1}_i. The two solutions are

$$I^{t+1}_i = V^t_i$$

and

$$I^{t+1}_i = I^t_i \frac{1 + \frac{V^t_i}{I^t_i} + \frac{V^t_i V^t_{i-1}}{I^t_i I^t_{i-1}} + \cdots + \frac{V^t_i V^t_{i-1} \cdots V^t_{i+2}}{I^t_i I^t_{i-1} \cdots I^t_{i+2}}}{1 + \frac{V^t_{i-1}}{I^t_{i-1}} + \frac{V^t_{i-1} V^t_{i-2}}{I^t_{i-1} I^t_{i-2}} + \cdots + \frac{V^t_{i-1} V^t_{i-2} \cdots V^t_{i+1}}{I^t_{i-1} I^t_{i-2} \cdots I^t_{i+1}}} \tag{2.9}$$

$$= V^t_i + I^t_i \frac{1 - \frac{V^t_1 V^t_2 \dots V^t_N}{I^t_1 I^t_2 \dots I^t_N}}{1 + \frac{V^t_{i-1}}{I^t_{i-1}} + \frac{V^t_{i-1} V^t_{i-2}}{I^t_{i-1} I^t_{i-2}} + \cdots + \frac{V^t_{i-1} V^t_{i-2} \cdots V^t_{i+1}}{I^t_{i-1} I^t_{i-2} \cdots I^t_{i+1}}}. \tag{2.10}$$

The first one does not satisfy condition (2.4). The other one gives the time evolution for I^t_i.

Now, we calculate the ultradiscrete limit of (2.6) and (2.10).

To obtain the ultradiscrete limit, we put $I^t_i = \exp\left[-\frac{Q^t_i + o(1)}{\varepsilon}\right]$, $V^t_i = \exp\left[-\frac{W^t_i + o(1)}{\varepsilon}\right]$, and take a limit $\varepsilon \to 0^+$. (The assumption $Q^t_j = \lim_{\varepsilon\to 0^+} -\varepsilon \log I^t_j(\varepsilon)$ is equivalent to $I^t_j(\varepsilon) = \exp\left[-\frac{Q^t_i + o(1)}{\varepsilon}\right]$, where $f(\varepsilon) = o(1) \Leftrightarrow \lim_{\varepsilon\to 0^+} f(\varepsilon) = 0$.) Note that one can define each $o(1)$ as a real function on condition that $\left\{I_j^t, V_j^t\right\}$ is a real positive solution. By virtue of the fact that

$$\lim_{\varepsilon\to 0^+} -\varepsilon \log\left(\mathrm{e}^{-(a+o_1(\varepsilon))/\varepsilon} + \mathrm{e}^{-(b+o_2(\varepsilon))/\varepsilon}\right) = \min[a, b], \qquad o_1(\varepsilon), o_2(\varepsilon) \in \mathbb{R} \tag{2.11}$$

and

$$0 \leqslant \lim_{\varepsilon\to 0^+} \frac{\prod_{i=1}^N V_i^t}{\prod_{i=1}^N I_i^t} < 1 \quad \Rightarrow \quad \lim_{\varepsilon\to 0^+} \varepsilon \log\left[1 - \frac{V_1^t V_2^t \ldots V_N^t}{I_1^t I_2^t \ldots I_N^t}\right] = 0,$$

it is a straightforward result that (2.6) yields (2.2), and (2.10) yields (2.1) if Q_i^t, W_i^t exist. □

Remark 2.1. (2.9) implies

$$I_j^0, V_j^0 \in \mathbb{R}_{>0}\ (j = 1, 2, \ldots, N) \quad \Rightarrow \quad I_j^t, V_j^t \in \mathbb{R}_{>0}\ (j = 1, 2, \ldots, N, t \in \mathbb{N}).$$

Remark 2.2. The assumption that $\left\{I_j^t, V_j^t\right\}$ is a real positive solution is essential to the proof of proposition 2.1. Otherwise equation (2.11) fails in a certain situation (see [2]).

We shall use this proposition to solve the initial value problem of the pBBS. Our strategy can be summarized as follows:

(i) For given initial data Q_j^0, W_j^0 $(j = 1, 2, \ldots, N)$, we associate initial values with the pd Toda equation as

$$Q_j^t = \lim_{\varepsilon\to 0^+} -\varepsilon \log I_j^t(\varepsilon), \qquad W_j^t = \lim_{\varepsilon\to 0^+} -\varepsilon \log V_j^t(\varepsilon). \tag{2.12}$$

(For example

$$I_j^0 = k_j \exp\left[-\frac{Q_j^0}{\varepsilon}\right], \qquad V_j^0 = k'_j \exp\left[-\frac{W_j^0}{\varepsilon}\right], \tag{2.13}$$

for positive $k_j, k'_j > 0$, $(j = 1, 2, \ldots, N)$.)

(ii) Then, we solve the initial value problem of the pd Toda equation by the inverse scattering method. The solution $\left\{I_j^t(\varepsilon), V_j^t(\varepsilon)\right\}$ depends on the parameter ε.

(iii) In principle, by proposition 2.1, the solution to the pBBS is obtained as

$$Q_j^t = \lim_{\varepsilon\to 0^+} -\varepsilon \log I_j^t(\varepsilon), \qquad W_j^t = \lim_{\varepsilon\to 0^+} -\varepsilon \log V_j^t(\varepsilon). \tag{2.14}$$

Remark 2.3. One does not always have to define I_j^0 and V_j^0 as (2.13). Indeed, one could define these numbers freely on condition (2.12).

2.2. Solution of the pd Toda equation

The initial value problem of the pd Toda equation was solved by the algebro-geometric method [1]. In this paper, we omit most of the details of the method and give only the solution.

Let C be a hyperelliptic curve of genus g, and define the base of $H_1(C, \mathbb{Z})$ as in figure 3. We denote the normalized 1-form on C by $\{\omega_i\}_{i=1}^g$, and the period matrix of C by $B = \left(\int_{b_i} \omega_j\right)_{i,j}$.

Remark 2.4. $\{\omega_i\}$ is normalized $\Leftrightarrow \int_{a_i} \omega_j = \delta_{i,j}\ \forall i, j$.

A hyperelliptic curve C of degree g can be expressed as

$$\mu^2 = G(\lambda),$$

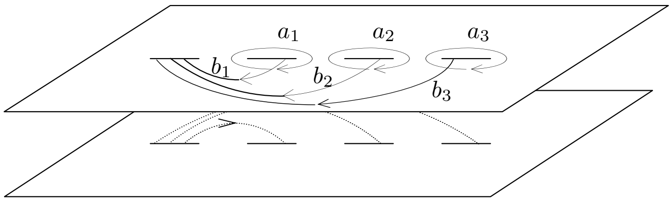

Figure 3. Canonical basis of $H_1(C, \mathbb{Z})$. (Case for $g = 3$.)

where $G(\lambda)$ is a polynomial in λ of degree $2g+1$ or $2g+2$. Any holomorphic differential on C can be rewritten as

$$c_{g-1}\frac{\lambda^{g-1}\,\mathrm{d}\lambda}{\mu} + c_{g-2}\frac{\lambda^{g-2}\,\mathrm{d}\lambda}{\mu} + \cdots + c_0\frac{\mathrm{d}\lambda}{\mu}, \qquad c_0, \ldots, c_{g-1} \in \mathbb{C}.$$

Let us define the complex constants $c_{j,k}$ $(j = 1, 2, \ldots, g, k = 0, 1, \ldots, g-1)$ as

$$\omega_j = c_{j,g-1}\frac{\lambda^{g-1}\,\mathrm{d}\lambda}{\mu} + c_{j,g-2}\frac{\lambda^{g-2}\,\mathrm{d}\lambda}{\mu} + \cdots + c_{j,0}\frac{\mathrm{d}\lambda}{\mu}.$$

The Abelian mapping and the theta function are the most important tools in the method. We define the quotient space $\mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$ obtained by the equivalence relation

$$x \sim y \in \mathbb{C}^g \Leftrightarrow x - y \in \mathbb{Z}^g + B\mathbb{Z}^g.$$

For a fixed point $P_0 \in C$, the mapping

$$C \ni P \mapsto \int_{P_0}^{P} \boldsymbol{\omega} \equiv \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g\right) \in \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$$

is a well-defined Abelian mapping. The Abelian mapping is usually denoted by

$$\boldsymbol{A}(P) = \int_{P_0}^{P} \boldsymbol{\omega}.$$

The Abelian mapping of a divisor $D = \sum_i n_i P_i$ is defined by the formula

$$\boldsymbol{A}(D) = \sum_i n_i \int_{P_0}^{P_i} \boldsymbol{\omega}.$$

Definition 2.2. *Let B be a $g \times g$ matrix which satisfies the relation*

$$B = B^t \qquad \textit{and} \qquad \operatorname{Im} B > 0.$$

Then the theta function $\theta(\boldsymbol{z}, B)$ for $\boldsymbol{z} \in \mathbb{C}^g$ is defined as the holomorphic function

$$\theta(\boldsymbol{z}, B) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp\left(\pi \mathrm{i} \boldsymbol{n}^t B \boldsymbol{n} + 2\pi \mathrm{i} \boldsymbol{n}^t \boldsymbol{z}\right).$$

Remark 2.5. The theta function $\theta(\boldsymbol{z}, B)$ satisfies

$$\theta(\boldsymbol{z} + \boldsymbol{e}_j, B) = \theta(\boldsymbol{z}, B), \qquad \boldsymbol{e}_j = (0, 0, \ldots, \overset{j}{\hat{1}}, \ldots, 0)^t$$

and

$$\theta(\boldsymbol{z} + \boldsymbol{b}_j, B) = \mathrm{e}^{-2\pi \mathrm{i} z_j - \pi \mathrm{i} B_{jj}} \theta(\boldsymbol{z}, B), \qquad \boldsymbol{b}_j = (B_{1j}, \ldots, B_{gj})^t.$$

In our algebro-geometric method (i.e. inverse scattering method), we use these functions and Abelian integrals on some hyperelliptic curve C to describe the solution of the pd Toda equation.

To define the curve C associated with the initial condition $\left\{I_j^0, V_j^0\right\}_{j=0}^{N-1}$, we prepare two sequences $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ by (2.15) and (2.16):

$$\begin{cases} x_{n+1} = \left\{\lambda - \left(I_{n+1}^0 + V_n^0\right)\right\}x_n - \left(I_n^0 V_n^0\right)x_{n-1} \\ y_{n+1} = \left\{\lambda - \left(I_{n+1}^0 + V_n^0\right)\right\}y_n - \left(I_n^0 V_n^0\right)y_{n-1} \end{cases}, \tag{2.15}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.16}$$

Let C be a hyperelliptic curve defined by

$$\mu^2 = \Delta(\lambda)^2 - 4m^2, \tag{2.17}$$

where

$$\Delta(\lambda) = x_{N+1} + y_N, \qquad m^2 = \prod_{i=1}^{N} I_i^0 V_i^0. \tag{2.18}$$

Note that $\Delta(\lambda)$ is a monic polynomial in λ of degree N. The genus of the hyperelliptic curve C is $N-1$ $(=: g)$.

Separately form the definition of C, we define $N-1$ complex numbers $\mu_j, (j = 1, 2, \ldots, N-1)$ as the roots of

$$y_{N+1} = 0. \tag{2.19}$$

Note that y_{N+1} is a polynomial of degree $N-1$, the highest coefficient of which is $-I_1^0 V_1^0$.

Theorem 2.2 ([1])**.** *Let C be a hyperelliptic curve (2.17), and define the canonical basis of $H_1(C,\mathbb{Z})$ as in figure 3. Then, the solution of the pd Toda equation (2.5) and (2.6) is expressed as follows:*

$$I_{n+2}^t + V_{n+1}^t = \sum_{j=0}^{g} \lambda_j - \sum_{j=1}^{g} \int_{a_j} \lambda\omega_j - \sum_{j=1}^{g} c_{j,g-1} \frac{\mathrm{d}}{\mathrm{d}u_j} \log \frac{\theta(n\boldsymbol{r} + t\boldsymbol{\nu} + \boldsymbol{c}; B)}{\theta((n+1)\boldsymbol{r} + t\boldsymbol{\nu} + \boldsymbol{c}; B)}, \tag{2.20}$$

where

$$\boldsymbol{r} = \boldsymbol{A}(\infty^- - \infty^+), \qquad \boldsymbol{\nu} = \boldsymbol{A}(0 - \infty^+), \qquad \boldsymbol{c}(0) = \boldsymbol{A}\left(\infty_+ - \sum_{j=1}^{g} P_j^0\right) - \boldsymbol{K}$$

∞^+ is the point at infinity in the upper sheet of C, and ∞^- is the point at infinity in the lower sheet. 0 is the point in the lower sheet with $\lambda(0) = 0$. $\boldsymbol{K}$ is a Riemann constant of C [3, 4], and $\lambda_0, \lambda_1, \ldots, \lambda_g$ are the roots of $\Delta(\lambda) = 0$. And P_j is a point on C, which satisfies $\lambda(P_j) = \mu_j$. The sign $\frac{\mathrm{d}}{\mathrm{d}u_j}$ means the differential for the jth component.

2.3. Young diagram

In this section, we briefly review the correspondence between box-ball systems and Young diagrams. A Young diagram is a collection of boxes as shown in figure 4. We define a Young diagram associated with a state of the pBBS.

Let us consider the pBBS which has N-solitons (section 2.1). When we regard the pBBS as a dynamical system of a finite sequence of 0's and 1's, we can introduce the following operation which we shall call '10-elimination'.

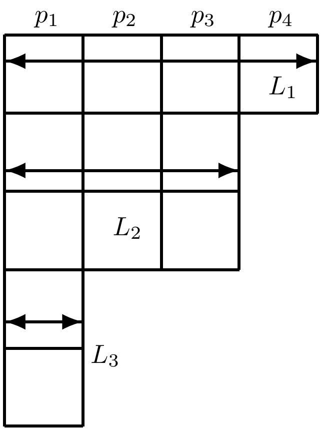

Figure 4. The Young diagram associated with the state in figure 5.

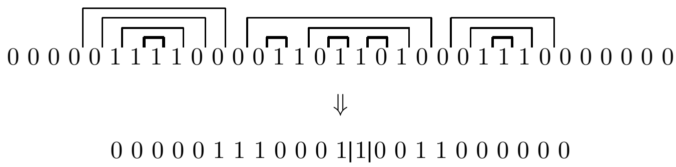

Figure 5. An example of 10-elimination. A 3-soliton system with two 0-solitons is obtained from a 5-soliton system.

(i) For a given state, connect all the ‘10’ pairs in the sequence with arc lines.
(ii) Neglecting the 10 pairs which were connected in the first step, connect all the remaining 10’s with arc lines.
(iii) Repeat the above procedure until all the 1’s are connected to 0’s.
(iv) Eliminate the 10’s in a state, and obtain a new sequence.

Figure 5 shows an example of 10-elimination. The mark ‘|’ means ‘0-soliton’ which has no entry but has a position. A 0-soliton appears when we eliminate a soliton of length 1. We can perform this ‘10-elimination’ repeatedly and transform any N-soliton system into a $(N-k)$-soliton system with k 0-solitons. Note that k is the number of the shortest solitons in the N-soliton state. Note also that the 0-solitons do not move under the time evolution rule.

Let p_1 be the number of 10 pairs in a state of the pBBS, connected with arc lines in the first step of 10-elimination (i.e. after one elimination). Similarly, we denote by p_j the number of 10 pairs in the jth step of 10-elimination. Note that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_l$, where l is the number of the last step. The most important aspect of these integers p_j is the fact that the series $\{p_1, p_2, \ldots, p_l\}$ are conserved quantities for the time evolution of the pBBS [5]. Using this series, we can associate a state of the pBBS with a Young diagram with p_j boxes in the jth column ($j = 1, 2, \ldots, l$) (see figure 4). Then let us denote the distinct lengths of the rows by $\{L_1, L_2, \ldots, L_s\}$. Note that $L_1 > L_2 > \cdots > L_s$.

The following is a main theorem in this paper.

Theorem 2.3. *Let $C : \mu^2 = \Delta(\lambda)^2 - 4m^2$ be the hyperelliptic curve associated with the initial value problem of the pd Toda equation defined by (2.5), (2.6), (2.7), (2.8). And define*

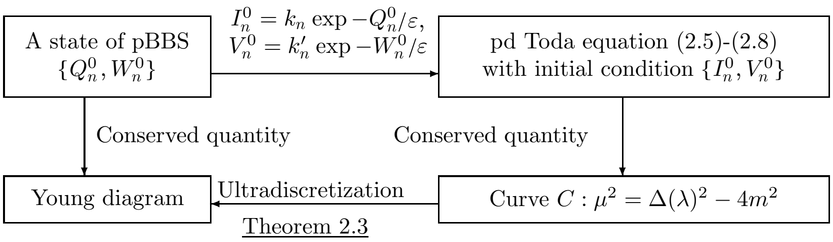

Figure 6. The relation between the conserved quantities of two systems, the pBBS and the pd Toda equation.

$I_n^0(\varepsilon)$, $V_n^0(\varepsilon)$, $(n = 1, 2, \ldots, N)$ *such that*

$$Q_n^0 = -\lim_{\varepsilon\to 0} \varepsilon \log I_n^0(\varepsilon), \qquad W_n^0 = -\lim_{\varepsilon\to 0} \varepsilon \log V_n^0(\varepsilon),$$

and all of the roots of $\Delta(\lambda)^2 - 4m^2 = 0$ *are simple. (It is possible to choose such* I_n^0 *and* V_n^0 *because the discriminant of* $\Delta(\lambda)^2 - 4m^2$ *is a non-trivial polynomial of* I_n^0, V_n^0*s (see remark 2.3)). Then all of these roots are positive. And these roots*

$$0 < \lambda_0^- < \lambda_0^+ < \lambda_1^- < \lambda_1^+ < \cdots < \lambda_g^- < \lambda_g^+, \qquad (g = N - 1)$$

satisfy

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log \lambda_j^\pm = \left\{ \begin{matrix} \text{the length of the } (N-j)\text{th row of the} \\ \text{Young diagram associated with the state } \{Q_n^0, W_n^0\} \end{matrix} \right\}.$$

By virtue of theorem 2.3, the ultradiscrete limits of the data which are described by $\lambda_j^\pm$ $(j = 0, 1, \ldots, g)$ can be expressed in terms of the Young diagram (figure 6). In fact, several essential data for the pd Toda equation—fundamental period etc—can be expressed by $\lambda_j^\pm$s only. Hence, we can express the ultradiscrete limit of these data by the information of the associated Young diagram.

3. Proof of theorem 2.3

3.1. Ultradiscrete limit

For convenience, we first state several lemmas which we will use in the rest of the present paper.

Definition 3.1. *Let* ε *be a positive number, and let* $f(\varepsilon)$ *be a continuous real-valued function of* ε*. Let us introduce the symbol '* $\twoheadrightarrow$ *' which means*

$$f \twoheadrightarrow F \Leftrightarrow \begin{cases} \exists\delta > 0 \quad s.t. \quad 0 < \varepsilon < \delta \Rightarrow f(\varepsilon) > 0 \\ F = \lim_{\varepsilon\to 0} \varepsilon \log f(\varepsilon). \end{cases}$$

Remark 3.1. By definition, $f \twoheadrightarrow F \Leftrightarrow \log f \sim F/\varepsilon(\varepsilon \to 0)$.

Definition 3.2. *The sign '* $\overset{\text{u}}{\sim}$ *' stands for the relation*

$$f \overset{\text{u}}{\sim} g \Leftrightarrow f \twoheadrightarrow F, \qquad g \twoheadrightarrow G \qquad \text{and} \qquad F = G.$$

Remark 3.2. The sign '$\overset{\mathrm{u}}{\sim}$' is not equivalent with the usual sign '$\sim$', i.e., $f \sim g \Leftrightarrow \exists m, M > 0$ s.t. $0 < m < \lim_{\varepsilon\to 0}|f(\varepsilon)/g(\varepsilon)| < M < +\infty$. For example, if $f(\varepsilon) = \varepsilon^{-1}, g(\varepsilon) = \varepsilon^{-2}$, then $f \overset{\mathrm{u}}{\sim} g$ but $f \nsim g$. Precisely speaking, under the condition $f(\varepsilon), g(\varepsilon) > 0$ for $0 < \varepsilon \ll 1$, $f \sim g \Rightarrow f \overset{\mathrm{u}}{\sim} g$ holds, but the inverse relation does not necessarily hold.

Lemma 3.1. *Let $f_1, f_2, \ldots, f_N, g$ be continuous real-valued functions of ε with $g = f_1 + f_2 + \cdots + f_N$. If*

$$g \rightarrowtriangle G, \qquad f_1 \rightarrowtriangle F_1, \ldots, f_N \rightarrowtriangle F_N,$$

then $G = \max\{F_1, \ldots, F_N\}$.

Proof. By definition, for some number $\delta > 0$, $f_1(\varepsilon), \ldots, f_N(\varepsilon), g(\varepsilon)$ are all positive if $\varepsilon \in (0, \delta)$. It can be assumed that $F_1 \geqslant F_2 \geqslant \cdots \geqslant F_N$ without loss of generality. For the largest numbers $F_1 = F_2 = \cdots = F_m$ $(1 \leqslant m \leqslant N)$, one can rearrange the index if necessary and assume

$$f_1 \succeq f_2 \succeq \cdots \succeq f_m,$$

where $h \succeq k \overset{\mathrm{Def}}{\Longleftrightarrow} \exists C > 0, \lim_{\varepsilon\to 0}|k(\varepsilon)/h(\varepsilon)| < C < +\infty$. So, $g \sim f_1$ is obvious, therefore, noticing remark 3.2 and definition 3.2, $G = F_1$ is proved. □

We have the following obvious lemma.

Lemma 3.2. *Let f, g be continuous real-valued functions of ε:*

$$f \rightarrowtriangle F, \qquad g \rightarrowtriangle G \quad \Rightarrow \quad fg \rightarrowtriangle F + G.$$

Lemma 3.3. *Let f, g, h be continuous real-valued functions of ε with $f(\varepsilon) = g(\varepsilon) + h(\varepsilon)$. If $f \rightarrowtriangle F$, $|g| \rightarrowtriangle G$, $|h| \rightarrowtriangle H$, and $G \neq H$, then $F = \max[G, H]$.*

Proof. Without loss of generality, one can assume $G > H$. Let δ be a positive number which admits

$$0 < \varepsilon < \delta \Rightarrow \begin{cases} f(\varepsilon) > 0 \\ |h(\varepsilon)| < C_\delta\,|g(\varepsilon)| \end{cases}$$

for some C_δ which depends on only δ and $C_\delta \to 0$ $(\delta \to 0)$.

Thus, $0 < f \leqslant |g| + |h| < (1 + C_\delta)\,|g|$ for $\varepsilon \in (0, \delta)$, and $F \leqslant G$ holds. On the other hand, the inequality $0 < |g| \leqslant |f| + |h|$ gives

$$1 \leqslant \frac{|f|}{|g|} + \frac{|h|}{|g|} < \frac{|f|}{|g|} + C_\delta \qquad \text{for} \quad \varepsilon \in (0, \delta).$$

As $C_\delta \to 0$ for decreasing δ, it follows that $F \geqslant G$. □

Remark 3.3. If we omit the condition '$G \neq H$', the claim of lemma 3.3 becomes

$$F \leqslant \max[G, H].$$

Lemma 3.4. *Let $f(\lambda, \varepsilon)$ be a polynomial in λ with real coefficients of degree $N + 1$:*

$$f(\lambda, \varepsilon) = \lambda^{N+1} - k_N(\varepsilon)\lambda^N + k_{N-1}(\varepsilon)\lambda^{N-1} - \cdots + (-1)^{N+1}k_0(\varepsilon)$$

where $k_j(\varepsilon) > 0$ $(j = 0, 1, \ldots, N)$ and $k_j \rightarrowtriangle K_j$. Then, the roots of the equation $f(\lambda, \varepsilon) = 0$, $\lambda_0(\varepsilon) < \lambda_1(\varepsilon) < \cdots < \lambda_N(\varepsilon)$ satisfy

$$\lambda_N \rightarrowtriangle K_N, \qquad \lambda_{N-1} \rightarrowtriangle K_{N-1} - K_N, \ldots, \lambda_0 \rightarrowtriangle K_0 - K_1.$$

Proof. The fundamental relation between roots and coefficients gives

$$k_N = \lambda_0 + \lambda_1 + \cdots + \lambda_N,$$
$$k_{N-1} = \lambda_0\lambda_1 + \cdots + \lambda_{N-1}\lambda_N,$$
$$\ldots,$$
$$k_0 = \lambda_0\lambda_1 \cdots \lambda_N.$$

Using lemmas 3.1 and 3.2, it is easy to prove the lemma. □

3.2. Ultradiscretization of the polynomial $\Delta(\lambda)$

In this subsection, we define and calculate the key parameters associated with an initial state of the pBBS denoted by U_j, ($j = 0, 1, \ldots, N-1$) and P_k, ($k = 0, 1, \ldots, N-2$). In the subsequent subsections, the ultradiscrete limit of the solution of the pd Toda equation (2.20) is expressed by U_j and P_k.

Let $C : \mu^2 = \Delta(\lambda)^2 - 4m^2$ be the hyperelliptic curve defined by (2.17). Note that $\Delta(\lambda)$ is a monic polynomial in λ of degree N $(= g+1)$.

We use the following two propositions without the proof.

Proposition 3.5. *The roots of*

$$\Delta(\lambda) = 0$$

are all real and positive.

Proof. The proof is given in [6]. □

Proposition 3.6. *If the equation*

$$\Delta(\lambda)^2 - 4m^2 = 0 \tag{3.1}$$

has only simple roots, all of these roots are real and positive.

Proof. The proof is given in [6]. □

Definition 3.3. *Let us denote $\Delta(\lambda)$ by*

$$\Delta(\lambda) = \lambda^{g+1} - u_g\lambda^g + u_{g-1}\lambda^{g-1} - \cdots + (-1)^{g+1}u_0.$$

We define the real numbers U_j $(j = 0, 1, \ldots, g)$ as

$$U_j := -\lim_{\varepsilon\to 0^+} \varepsilon \log u_j,$$

or equivalently, $u_j \to\!\!\!\triangleright\ -U_j$.

To define P_k, let us consider the polynomial $y_{N+1}(\lambda)$ (2.19). Note that $y_{N+1}(\lambda)$ is a polynomial of degree $N-1$ and the roots of $y_{N+1}(\lambda) = 0$ are μ_k $(k = 1, 2, \ldots, g)$.

Remark 3.4. The roots of $y_{N+1}(\lambda) = 0$ are usually called the auxiliary spectrum. It is known that all auxiliary spectra are real and positive [6].

Definition 3.4. *Let us denote $y_{N+1}(\lambda)$ as*

$$y_{N+1} = -I_1^0 V_1^0 \left\{\lambda^g - v_{g-1}\lambda^{g-1} + v_{g-2}\lambda^{g-2} + \cdots + (-1)^g v_0\right\}$$

(see (2.19)). We define the real numbers P_k $(k = 0, 1, \ldots, g-1)$, as

$$P_k := -\lim_{\varepsilon\to 0^+} \varepsilon \log v_k,$$

or equivalently, $v_k \to\!\!\!\triangleright\ -P_k$.

To calculate U_j, we need to prepare a few notations.

Let the set $\mathcal{A}(N)$ be

$$\mathcal{A}(N) := \left\{ \begin{matrix} \{a_1-1, a_1, \ldots, a_k-1, a_k\} \\ \in 2^{\mathbb{Z}/N\mathbb{Z}} \end{matrix} \middle| \begin{matrix} k = 1, 2, \ldots, \left[\frac{N}{2}\right] \\ \forall i, j, a_i \not\equiv a_j - 1 \wedge i \neq j \Rightarrow a_i \not\equiv a_j \end{matrix} \right\} \cup \{\emptyset\}.$$

An element of $\mathcal{A}(N)$ is a subset of $\mathbb{Z}/N\mathbb{Z}$, which consists of pairs of two consecutive numbers. (In $\mathbb{Z}/N\mathbb{Z}$, we regard N and 1 are consecutive numbers.) For $N \geqslant 3$, the number of elements of $\mathcal{A}(N)$ is equal to $F_N + F_{N-2}$, where F_N is the Nth Fibonacci number. ($F_{N+2} = F_{N+1} + F_N$, $F_1 = 1$, $F_2 = 2$.)

Proposition 3.7. *The polynomial* $\Delta(\lambda) = x_{N+1} + y_N$ *is expressed as*

$$\Delta(\lambda) = \sum_{(j_1-1, j_1, \ldots, j_k-1, j_k) \in \mathcal{A}(N)} Y_{j_1} \ldots Y_{j_k} X_{i_1} \ldots X_{i_{N-2k}}$$

where $\{j_1 - 1, j_1, \ldots, j_k - 1, j_k\} \sqcup \{i_1, \ldots, i_{N-2k}\} = \mathbb{Z}/N\mathbb{Z}$, *and* $Y_j = -I_j V_j$, $X_i = \lambda - (I_{i+1} + V_i)$.

The proof of proposition 3.7 is elementary though slightly involved. We therefore defer the proof to the appendix. Defining $a_i := I_{i+1} + V_i$ and $b_i := I_i V_i$, we find

Proposition 3.8. *The coefficients of the polynomial*

$$\Delta(\lambda) = \lambda^{g+1} - u_g \lambda^g + u_{g-1} \lambda^{g-1} - \cdots + (-1)^{g+1} u_0$$

satisfy

$$u_0 \rightarrowtriangle -(Q_1 + Q_2 + \cdots + Q_N) \tag{3.2}$$

and

$$u_g \rightarrowtriangle -\min[Q_i, W_i]. \tag{3.3}$$

Equivalently, $U_0 = Q_1 + Q_2 + \cdots + Q_N$ *and* $U_g = \min[Q_i, W_i]$.

Proof. It is sufficient to prove

$$u_0 = \sum_{(j_1-1, j_1, \ldots, j_k-1, j_k) \in \mathcal{A}(N)} (-b_{j_1}) \ldots (-b_{j_k}) a_{i_1} \ldots a_{i_{N-2k}} \tag{3.4}$$

$$= I_1 I_2 \cdots I_N + V_1 V_2 \cdots V_N \tag{3.5}$$

and

$$u_g = I_1 + I_2 + \cdots + I_N + V_1 + V_2 + \cdots + V_N. \tag{3.6}$$

(3.6) is a direct consequence of proposition 3.7. It remains to prove (3.5). Substituting $a_i = I_{i+1} + V_i$ and $b_i = I_i V_i$ to (3.4), two types of terms appear, namely those that contain $V_k I_k$ for some k (type (i)), and those that do not (type (ii)).

Among all the terms in (3.4), a contribution '$V_k I_k$' must come from the term which contains $-b_k$ or $a_{k-1} a_k$ only. For any term which contains $-b_k$, there exists one term in which $-b_k$ is replaced by $a_{k-1} a_k$ in (3.4). Hence we can conclude that the summation of all terms of type (i) will cancel out. The only terms of type (ii) are $I_1 I_2 \cdots I_N$ and $V_1 V_2 \cdots V_N$ because the term of this type must come from $a_1 a_2 \cdots a_N$. □

The ultradiscrete limit of $u_1, u_2, \ldots, u_{g-1}$ are also obtained in a similar manner. Let

$$X = \{A_i | A_{2l-1} = Q_l, A_{2l} = W_l, \ (l = 1, 2, \ldots, N)\}$$

and

$$\mathcal{B}(k,N)=\left\{\{A_{\sigma(i)}\}\subset X\;\middle|\;\begin{array}{c}1\leqslant\sigma(1)<\sigma(2)<\cdots<\sigma(N-k)\leqslant 2N\\ \sigma(i)+1<\sigma(i+1),\forall i\\ \sigma(1)=1\Rightarrow\sigma(N-k)\neq 2N\end{array}\right\}.$$

Proposition 3.9. *It follows that*

$$u_k \longrightarrow -\min_{\{A_{\sigma(i)}\}\in\mathcal{B}(k,N)}\left\{\sum_{1\leqslant i\leqslant N-k}A_{\sigma(i)}\right\}(\equiv -U_k).$$

Proof. From the proof of proposition 3.8, u_k can be obtained in the following way. First, calculate

$$\sum_{\{i_1,i_2,\ldots,i_{N-k}\}\subset\mathbb{Z}/N\mathbb{Z}} a_{i_1}a_{i_2}\ldots a_{i_{N-k}}. \tag{3.7}$$

And pick up the terms which contain no V_lI_ls. Since $a_l=I_{l+1}+V_l$, a term that contains V_lI_{l+1} cannot exist in (3.7). Conversely, a term of length $N-k$ which has neither V_lI_l nor V_lI_{l+1} necessarily appears in (3.7). □

We can calculate $P_0,P_1,\ldots,P_{g-1}$ analogously.

Proposition 3.10. *The polynomial $y_{N+1}(\lambda)$ is of the form*

$$y_{N+1}(\lambda)=-b_1\times\sum_{(j_1-1,j_1,\ldots,j_k-1,j_k)\in\mathcal{A}'(N)}Y_{j_1}\ldots Y_{j_k}X_{i_1}\ldots X_{i_{N-1-2k}},$$

where X and Y are given in proposition 3.7, and $\mathcal{A}'(N)=\mathcal{A}(N)\cap 2^{(\mathbb{Z}/N\mathbb{Z}-\{1\})}$ is a subset of $\mathcal{A}(N)$ of which element $(j_1-1,j_1,\ldots,j_k-1,j_k)$ does not contain the number '1' $\in\mathbb{Z}/N\mathbb{Z}$.

We will prove this proposition in the appendix.

In the same way as in proposition 3.8 and proposition 3.9 we obtain:

Proposition 3.11. *It follows that*

$$v_k \longrightarrow -\min_{\{A_{\sigma(i)}\}\in\mathcal{B}'(k,N)}\left\{\sum_{1\leqslant i\leqslant N-k-1}A_{\sigma(i)}\right\}(\equiv -P_k),$$

where $\mathcal{B}'(k,N)=\mathcal{B}(k,N)\cap 2^{(X-\{Q_2,W_1\})}$ is a subset of $\mathcal{B}(k,N)$ which does not contain Q_2 or W_1.

Proof. As in the proof of proposition 3.9, v_k can be expressed as follows:

$$v_k=\left[\sum_{\{i_1,i_2,\ldots,i_{N-1-k}\}\subset(\mathbb{Z}/N\mathbb{Z}-\{1\})}a_{i_1}a_{i_2}\cdots a_{i_{N-1-k}}\right]_{V_lI_l\to 0}.$$

Since $a_1=I_2+V_1$, we obtain the proposition. □

The following lemma can be obtained from proposition 3.9 by combinatorial arguments, which we will give in the appendix. Let $\mathcal{Z}_N$ be a set of an N-soliton state of the pBBS.

Lemma 3.12. *Let $x\in\mathcal{Z}_N$ be an N-soliton state, and let $U_0,U_1,\ldots,U_g$ be the positive integers defined as in definition 3.3. Then U_k is equal to the number of boxes below the kth row in the Young diagram (see figure 7).*

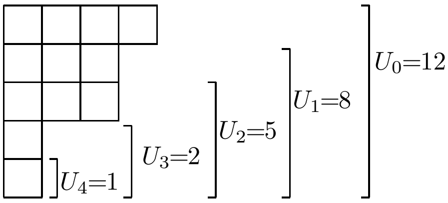

Figure 7. Example of the interpretation of the $U_k's$ of lemma 3.12.

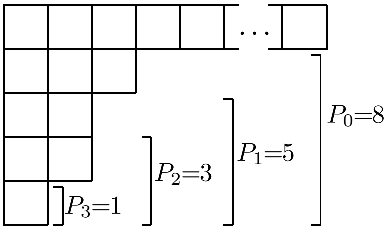

Figure 8. Example of a Young diagram corresponding to a state y.

Corollary 3.13. *Let $x \in \mathcal{Z}_N$ be an N-soliton state. We can obtain the integers $P_0, P_1, \ldots, P_{g-1}$ in definition 3.4 by the following procedure (see figure 8).*

(i) Let $Q_j^0(x), W_j^0(x)$ $(j = 0, 1, \ldots, g)$ be the integers defined in section 2.1 for a state $x \in \mathcal{Z}_N$. Consider another N-soliton state $y \in \mathcal{Z}_N$, with

$$\begin{aligned} &Q_j^0(y) = Q_j^0(x) && \text{for} \quad j \neq 2, \\ &W_i^0(y) = W_i^0(x) && \text{for} \quad i \neq 1, \\ &Q_2^0(y) \gg 1, && \text{and} \quad W_1^0(y) \gg 1. \end{aligned}$$

(ii) Let p_l be the length of lth row of the Young diagram corresponding to $y \in \mathcal{Z}_N$. Then,

$$P_j = \sum_{l=j+2}^{g-1} p_l.$$

Proof. From propositions 3.9 and 3.11, it is obvious that

$$\lim_{I_2, V_1 \to 0^+} u_{j+1} = v_j.$$

Then,

$$P_j = -\lim_{\varepsilon \to 0^+} \varepsilon \log v_j = -\lim_{\varepsilon \to 0^+} \lim_{I_2, V_1 \to 0^+} u_{j+1} = \lim_{I_2, V_1 \to 0^+} U_{j+1}.$$

The fact $I_2, V_1 \to 0^+ \Leftrightarrow Q_2, W_1 \to \infty$ completes the proof. □

3.3. Ultradiscretization of the curve $\mu^2 = \Delta(\lambda)^2 - 4m^2$

To complete the proof of theorem 2.3, we consider the asymptotic behaviour of the Riemann surface $C : \mu^2 = \Delta(\lambda)^2 - 4m^2$ when $\varepsilon \to 0$. From (2.18), we easily obtain that $m^2 = \mathrm{e}^{-\frac{L}{\varepsilon}}$, or equivalently

$$m \twoheadrightarrow -L/2, \tag{3.8}$$

where L is the number of boxes in the pBBS (see (2.18)).

Recall that we have denoted the roots of $\Delta(\lambda)^2 - 4m^2 = 0$ by

$$0 < \lambda_0^- < \lambda_0^+ < \lambda_1^- < \lambda_1^+ < \cdots < \lambda_g^- < \lambda_g^+$$

and the roots of $\Delta(\lambda) = 0$ by

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_g.$$

Note that $\lambda_j^- < \lambda_j < \lambda_j^+$ $(j = 0, 1, \ldots, g)$.

It is not easy to calculate the asymptotic behaviour of an Abelian integral on a general Riemann surface. However, as we shall see below, the problem of describing the asymptotic behaviour of C can be reduced to that of the degenerate case

$$C_0 : \mu^2 = \Delta(\lambda)^2.$$

By definition, it follows that

$$\Delta(\lambda)^2 - 4m^2 = \left(\lambda - \lambda_0^-\right)\left(\lambda - \lambda_0^+\right)\cdots\left(\lambda - \lambda_g^-\right)\left(\lambda - \lambda_g^+\right), \tag{3.9}$$

$$\Delta(\lambda) = (\lambda - \lambda_0)\cdots(\lambda - \lambda_g). \tag{3.10}$$

Clearly, equation (3.9) can be decomposed:

$$\Delta(\lambda) + 2m = \prod_{j=0}^{g}\left(\lambda - \lambda_j^{\sigma(j)}\right) \tag{3.11}$$

and

$$\Delta(\lambda) - 2m = \prod_{j=0}^{g}\left(\lambda - \lambda_j^{-\sigma(j)}\right) \tag{3.12}$$

where

$$\sigma(j) = \begin{cases} +(j = g-1, g-3, \ldots) \\ -(j = g, g-2, g-4, \ldots) \end{cases}$$

and $-\sigma(j)$ denotes the opposite sign to $\sigma(j)$.

By (3.10), (3.11) and (3.12), we have

$$\lambda_j - \lambda_j^{\sigma(j)} = \frac{-2m}{\prod_{k\neq j}\left(\lambda_j - \lambda_k^{\sigma(k)}\right)}, \tag{3.13}$$

$$\lambda_j - \lambda_j^{-\sigma(j)} = \frac{2m}{\prod_{k\neq j}\left(\lambda_j - \lambda_k^{-\sigma(k)}\right)}. \tag{3.14}$$

Lemma 3.14. *It follows that*

$$\lambda_j \overset{\mathrm{u}}{\sim} \lambda_j^{\pm}.$$

Proof. Let u_0 be the constant term of the polynomial $\Delta(\lambda)$ (see definition 3.3). Since $Q_1 + Q_2 + \cdots + Q_N < L/2$ (see section 2.1), we obtain

$$u_0 \overset{\mathrm{u}}{\sim} (u_0 \pm 2m),$$

from lemma 3.3, proposition 3.8, and (3.8). The proof then follows from (3.9), (3.10) and lemma 3.4. □

As a corollary of lemmas 3.4, 3.12 and 3.14, we obtain theorem 2.3.

The following proposition is used to calculate the ultradiscrete limit which is expressed as the difference of λ_js.

Proposition 3.15. *Under the condition*

$$j \neq k \quad \Rightarrow \quad U_{j+1} - U_j \neq U_{k+1} - U_k, \tag{3.15}$$

the ultradiscrete limit of $\left|\lambda_j - \lambda_k^{\pm}\right|$ *satisfies*

$$\begin{cases} \left|\lambda_j - \lambda_k^{\pm}\right| \twoheadrightarrow U_{j+1} - U_j & (j > k), \quad \text{and} \\ \left|\lambda_j - \lambda_k^{\pm}\right| \twoheadrightarrow U_{k+1} - U_k & (j < k), \end{cases}$$

where U_j $(j = 0, 1, \ldots, g)$ *is the real number defined by definition 3.3.*

Proof. From lemmas 3.4 and 3.14, it follows that

$$\lambda_j \twoheadrightarrow U_{j+1} - U_j, \qquad \text{and} \qquad \lambda_k^{\pm} \twoheadrightarrow U_{k+1} - U_k. \tag{3.16}$$

The assertion is proved immediately by virtue of lemma 3.3 and (3.15). □

Remark 3.5. The claim of proposition 3.15 is also true without condition (3.15). We can prove this assertion by using the fact that $U_{j+1} - U_j$ $(j = 0, 1, \ldots, g)$ can be perturbed independently over the real numbers, as we can naturally extend the domain of the initial condition of the pBBS $\left\{Q_n^0, W_n^0\right\} \subset \mathbb{N} \subset \mathbb{R}_{>0}$. The continuity of the time evolution rule of the pBBS ((2.1), (2.2), (2.3)) justifies the argument using such small perturbations.

4. Ultradiscretization of the solution of the pd Toda equation

4.1. Ultradiscretization of Abelian integrals

It is not easy to describe the normalized holomorphic differential of a Riemann surface in the general case. However, it is easy in the case of the degenerate curve C_0. In fact, the normalized holomorphic differential of C_0 is expressed as

$$\omega_j^0 = \frac{1}{2\pi \mathrm{i}} \left\{ \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_0} \right\} \mathrm{d}\lambda, \qquad (j = 1, 2, \ldots, g).$$

Theorem 4.1. *It follows that*

$$\int_{b_i} \omega_j \sim \int_{b_i} \omega_j^0, \qquad (\varepsilon \to 0).$$

To prove theorem 4.1, we prepare several lemmas. Let $\left\{\tilde{\omega}_j^0\right\}_{j=1}^g$ be holomorphic differentials on C defined by

$$\tilde{\omega}_j^0 := \frac{1}{2\pi \mathrm{i}} \frac{\Delta(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} \left\{ \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_0} \right\} \mathrm{d}\lambda.$$

In the first place, let us prove

Lemma 4.2. *It follows that*

$$\int_{b_i} \tilde{\omega}_j^0 \sim \int_{b_i} \omega_j^0. \tag{4.1}$$

Proof. Let X_i, $(i = 0, 1, \ldots, g-1)$ be real numbers which satisfy

$$\lambda_i^+ < X_i < \lambda_{i+1}^-, \qquad \begin{cases} \left(X_i - \lambda_i^+\right) \overset{\mathrm{u}}{\sim} X_i, \\ (\lambda_{i+1}^- - X_i) \overset{\mathrm{u}}{\sim} \lambda_{i+1}^-, \end{cases} \tag{4.2}$$

for example $X_i := \sqrt{\lambda_i^+ \lambda_{i+1}^-}$. Then we obtain

$$\int_{\lambda_i^+}^{X_i} \tilde{\omega}_i^0 = \frac{1}{2\pi \mathrm{i}} \int_{\lambda_i^+}^{X_i} \frac{\Delta(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} \left\{ \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_0} \right\} \mathrm{d}\lambda \tag{4.3}$$

$$= \frac{1}{2\pi \mathrm{i}} \left\{ \int_{\lambda_i^+}^{X_i} \frac{\prod_{k \neq i} (\lambda - \lambda_k)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} \mathrm{d}\lambda - \int_{\lambda_i^+}^{X_i} \frac{\prod_{k \neq 0} (\lambda - \lambda_k)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} \mathrm{d}\lambda \right\}. \tag{4.4}$$

Under the condition $\lambda_i^+ < \lambda < X_i$,

$$\left| \frac{\prod_{k \neq i} (\lambda - \lambda_k)}{\sqrt{\prod_{k \neq i} \left(\lambda - \lambda_k^-\right)\left(\lambda - \lambda_k^+\right)}} \right| = \left| \sqrt{\prod_{k \neq i} \frac{\lambda - \lambda_k}{\lambda - \lambda_k^-} \cdot \frac{\lambda - \lambda_k}{\lambda - \lambda_k^+}} \right| \tag{4.5}$$

$$= \left| \sqrt{\prod_{k \neq i} \left\{ 1 + \frac{\lambda_k^- - \lambda_k}{\lambda - \lambda_k^-} \right\} \cdot \left\{ 1 + \frac{\lambda_k^+ - \lambda_k}{\lambda - \lambda_k^+} \right\}} \right| \tag{4.6}$$

$$\overset{\mathrm{u}}{\sim} \left| \sqrt{\prod_{k \neq i} \left\{ 1 + \frac{\lambda_k^- - \lambda_k}{\lambda_i - \lambda_k} \right\} \cdot \left\{ 1 + \frac{\lambda_k^+ - \lambda_k}{\lambda_i - \lambda_k} \right\}} \right| \tag{4.7}$$

$$\overset{\mathrm{u}}{\sim} \left| \sqrt{\prod_{k \neq i} \{1 + e^{-I_k^-/\varepsilon}\} \cdot \{1 + e^{-I_k^+/\varepsilon}\}} \right|, \tag{4.8}$$

where

$$I_k^+, I_k^- > 0. \tag{4.9}$$

(The existence of positive numbers I_k^+, I_k^- in (4.9) is proved from proposition 3.8, proposition 3.9 and (3.8), and the fact

$$Q_0^0 + Q_1^0 + \cdots + Q_g^0 < \frac{L}{2}.$$

For example, we can show

$$\left| \frac{\lambda_k^- - \lambda_k}{\lambda_i - \lambda_k} \right| < \frac{2m}{\lambda_0 \lambda_1 \cdots \lambda_g}, \qquad (\varepsilon \ll 1)$$

$$\twoheadrightarrow -\frac{L}{2} + \left(Q_0^0 + Q_1^0 + \cdots + Q_g^0\right)$$

$$< 0,$$

which assures the existence of $I_k^- > 0$.)

Then there exist a positive number $B' > 0$ such that

$$(4.4) \quad \sim \frac{1 + O(\mathrm{e}^{-B'/\varepsilon})}{2\pi \mathrm{i}} \int_{\lambda_i^+}^{X_i} \left\{ \frac{1}{\sqrt{\left(\lambda - \lambda_i^-\right)\left(\lambda - \lambda_i^+\right)}} - \frac{1}{\sqrt{\left(\lambda - \lambda_0^-\right)\left(\lambda - \lambda_0^+\right)}} \right\} \mathrm{d}\lambda$$

$$\sim \frac{1+O(\mathrm{e}^{-B'/\varepsilon})}{2\pi\mathrm{i}}\left\{2\log\left[\sqrt{\frac{X_i-\lambda_i^+}{\lambda_i^+-\lambda_i^-}}+\sqrt{\frac{X_i-\lambda_i^-}{\lambda_i^+-\lambda_i^-}}\right]\right.$$
$$\left.-2\log\left[\sqrt{\frac{X_i-\lambda_0^+}{\lambda_0^+-\lambda_0^-}}+\sqrt{\frac{X_i-\lambda_0^-}{\lambda_0^+-\lambda_0^-}}\right]+2\log\left[\sqrt{\frac{\lambda_i^+-\lambda_0^+}{\lambda_0^+-\lambda_0^-}}+\sqrt{\frac{\lambda_i^+-\lambda_0^-}{\lambda_0^+-\lambda_0^-}}\right]\right\}$$
$$\sim \frac{1}{2\pi\mathrm{i}}\log\frac{X_i-\lambda_i}{X_i-\lambda_0}\frac{\lambda_i^+-\lambda_0}{\lambda_i^+-\lambda_i}=\int_{\lambda_i^+}^{X_i}\omega_i^0. \tag{4.10}$$

In the case of $i\neq j$, it is easy to show

$$\int_{\lambda_i^+}^{X_i}\tilde{\omega}_j^0\sim\int_{\lambda_i^+}^{X_i}\omega_j^0. \tag{4.11}$$

In a similar manner, it follows that

$$\int_{X_i}^{\lambda_i^-}\tilde{\omega_j}^0\sim\int_{X_i}^{\lambda_i^-}\omega_j^0. \tag{4.12}$$

Equations (4.10), (4.11) and (4.12) complete the proof. □

Next, we prove the following lemma, which completes the proof of theorem 4.1.

Lemma 4.3. *It follows that*

$$\int_{b_i}\omega_j\sim\int_{b_i}\tilde{\omega}_j^0.$$

Proof. Since ω_j^0 is a holomorphic differential on C, the Riemann bilinear equation [3] gives

$$\sum_{i=1}^{g}\left(A_{j'i}B_{ji}^0-A_{ji}^0B_{j'i}\right)=0,$$

where

$$A_{ji}^0=\int_{a_i}\tilde{\omega}_j^0,\qquad B_{ji}^0=\int_{b_i}\tilde{\omega}_j^0,\qquad A_{j'i}=\int_{a_i}\omega_{j'},\qquad B_{j'i}=\int_{b_i}\omega_{j'}.$$

Then we obtain

$$B_{jj'}^0=\sum_{i=1}^{g}A_{ji}^0B_{j'i}. \tag{4.13}$$

If $i\neq j$,

$$\left|A_{ij}^0\right|=\left|\frac{2}{2\pi\mathrm{i}}\int_{\lambda_i^-}^{\lambda_i^+}\left\{\frac{\prod_{k\neq j}(\lambda-\lambda_k)}{\sqrt{\Delta(\lambda)^2-4m^2}}-\frac{\prod_{k\neq 0}(\lambda-\lambda_k)}{\sqrt{\Delta(\lambda)^2-4m^2}}\right\}\mathrm{d}\lambda\right| \tag{4.14}$$

$$<\frac{1+O(\mathrm{e}^{-B''/\varepsilon})}{\pi}\left\{\left|\int_{\lambda_i^-}^{\lambda_i^+}\frac{(\lambda-\lambda_i)\,\mathrm{d}\lambda}{\sqrt{(\lambda-\lambda_i^-)(\lambda-\lambda_i^+)(\lambda-\lambda_k^-)(\lambda-\lambda_k^+)}}\right|\right.$$
$$\left.+\left|\int_{\lambda_i^-}^{\lambda_i^+}\frac{(\lambda-\lambda_i)\,\mathrm{d}\lambda}{\sqrt{(\lambda-\lambda_i^-)(\lambda-\lambda_i^+)(\lambda-\lambda_0^-)(\lambda-\lambda_0^+)}}\right|\right\} \tag{4.15}$$

$$\overset{\mathrm{u}}{\sim}\frac{\left|\lambda_i^+-\lambda_i^-\right|}{|\lambda_i-\lambda_k|+|\lambda_i-\lambda_0|} \tag{4.16}$$

$$\overset{\mathrm{u}}{\sim} \mathrm{e}^{-B'''/\varepsilon}, \tag{4.17}$$

where

$$B'' > 0 \qquad \text{and} \qquad B''' > 0. \tag{4.18}$$

Relation (4.18) can be shown in a way similar to (4.4). In the case of $i = j$,

$$A_{ii}^0 = \frac{1}{\pi \mathrm{i}} \int_{\lambda_i^-}^{\lambda_i^+} \left\{ \frac{\prod_{k\neq i}(\lambda - \lambda_k)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} - \frac{\prod_{k\neq 0}(\lambda - \lambda_k)}{\sqrt{\Delta(\lambda)^2 - 4m^2}} \right\} \mathrm{d}\lambda \tag{4.19}$$

$$= \frac{1 + O(\mathrm{e}^{-F/\varepsilon})}{\pi \mathrm{i}} \int_{\lambda_i^-}^{\lambda_i^+} \frac{\mathrm{d}\lambda}{\sqrt{(\lambda - \lambda_i^-)(\lambda - \lambda_i^+)}} \sim 1, \tag{4.20}$$

where $F > 0$. Substituting (4.17) and (4.20) to (4.13), then

$$B_{jj'}^0 \sim B_{j'j} = B_{jj'}.$$

□

By theorem 4.1, we can calculate the asymptotic behaviour in the limit $\varepsilon \to 0$ of the period matrix $B = (B_{ij})_{1\leqslant i,j\leqslant g}$ associated with the hyperelliptic curve $C : \mu^2 = \Delta(\lambda)^2 - 4m^2$:

$$B_{ij} \sim \int_{b_i} \omega_j^0 = 2 \int_{\lambda_0^+}^{\lambda_i^-} \frac{1}{2\pi \mathrm{i}} \left\{ \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_0} \right\} \mathrm{d}\lambda \tag{4.21}$$

$$= \frac{1}{\pi \mathrm{i}} \left[\log \frac{\lambda_i^- - \lambda_j}{\lambda_i^- - \lambda_0} \cdot \frac{\lambda_0^+ - \lambda_0}{\lambda_0^+ - \lambda_j} \right]. \tag{4.22}$$

Other parameters can be calculated similarly:

$$\nu_j \sim \int_{\infty^+}^0 \omega_j^0 = \int_{-\infty}^0 \frac{1}{2\pi \mathrm{i}} \left\{ \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_0} \right\} \mathrm{d}\lambda \tag{4.23}$$

$$= \frac{1}{2\pi \mathrm{i}} \log \frac{\lambda_j}{\lambda_0}, \tag{4.24}$$

$$(\boldsymbol{r})_j \sim \int_{\infty^+}^{\infty^-} \omega_j^0 = 2 \int_{-\infty}^{\lambda_0^-} \omega_j^0 = \frac{1}{\pi \mathrm{i}} \log \frac{\lambda_0^- - \lambda_j}{\lambda_0^- - \lambda_0}, \tag{4.25}$$

$$\int_{\lambda_0^+}^{\infty^+} \omega_j^0 = -\int_{-\infty}^{\lambda_0^+} \omega_j^0 = -\frac{1}{2\pi \mathrm{i}} \log \frac{\lambda_0^+ - \lambda_j}{\lambda_0^+ - \lambda_0}, \tag{4.26}$$

$$\sum_{i=1}^g \int_{\lambda_0^+}^{\mu_i} \omega_j^0 + k_j = \sum_{i=1}^g \left\{ \int_{\lambda_0^+}^{\lambda_i^-} + \int_{\lambda_i^-}^{\mu_i} \right\} \omega_j^0 + k_j \tag{4.27}$$

$$= \sum_{i=1}^g \left\{ \frac{1}{2} \int_{b_i} + \int_{\lambda_i^-}^{\mu_i} + \frac{1}{2} \sum_{l=1}^{i-1} \int_{a_l} \right\} \omega_j^0 + k_j. \tag{4.28}$$

Using the formula for the Riemann constant corresponding to the hyperelliptic curve

$$k_j = -\frac{1}{2} \sum_{i=1}^g B_{ji} + \frac{g + 1 - j}{2}, \tag{4.29}$$

(4.28) becomes

$$\sum_{i=1}^{g}\int_{\lambda_0^+}^{\mu_i}\omega_j^0+k_j=\sum_{i=1}^{g}\left(-\int_{\mu_i}^{\lambda_i^-}\omega_j^0+\frac{1}{2}\right)$$

$$=\sum_{i=1}^{g}\left[-\frac{1}{2\pi \mathrm{i}}\int_{\mu_i}^{\lambda_i^-}\left\{\frac{1}{\lambda-\lambda_j}-\frac{1}{\lambda-\lambda_0}\right\}\mathrm{d}\lambda+\frac{1}{2}\right] \tag{4.30}$$

$$=\sum_{i=1}^{g}\left[-\frac{1}{2\pi \mathrm{i}}\log\frac{\lambda_i^- -\lambda_j}{\lambda_i^- -\lambda_0}\cdot\frac{\mu_i-\lambda_0}{\mu_i-\lambda_j}+\frac{1}{2}\right]. \tag{4.31}$$

Now, using remark 3.1, proposition 3.15 and (3.13),

$$B_{jj}\sim\frac{1}{\pi \mathrm{i}}\log\frac{4m^2}{(\lambda_j-\lambda_0)^2\prod_{k\geqslant 1}(\lambda_k-\lambda_0)\prod_{0\leqslant k<j}(\lambda_j-\lambda_k)\prod_{j<k\leqslant g}(\lambda_k-\lambda_j)}$$

$$\sim\frac{1}{\pi \mathrm{i}}\frac{1}{\varepsilon}(2M-2(U_{j+1}-U_j)-\sum_{k\geqslant 1}(U_{k+1}-U_k)-j(U_{j+1}-U_j)-\sum_{j<k\leqslant g}(U_{k+1}-U_k))$$

$$=\frac{1}{\pi \mathrm{i}}\frac{1}{\varepsilon}(2M-(j+1)U_{j+1}+(j+2)U_j+U_1), \tag{4.32}$$

and for $i>j$

$$B_{ij}\sim\frac{1}{\pi \mathrm{i}}\log\frac{2m(\lambda_i-\lambda_j)}{(\lambda_j-\lambda_0)(\lambda_i-\lambda_0)\prod_{k\geqslant 1}(\lambda_k-\lambda_0)} \tag{4.33}$$

$$\sim\frac{1}{\pi \mathrm{i}}\frac{1}{\varepsilon}(M-(U_{j+1}-U_j)-\sum_{k\geqslant 1}(U_{k+1}-U_k)) \tag{4.34}$$

$$=\frac{1}{\pi \mathrm{i}}\frac{1}{\varepsilon}(M-U_{j+1}+U_j+U_1), \tag{4.35}$$

where $m\twoheadrightarrow M\ (=-L/2)$ and $U_{g+1}=0$.

Similarly,

$$\nu_j\sim\frac{1}{2\pi \mathrm{i}}\frac{1}{\varepsilon}(U_{j+1}-U_j-(U_1-U_0)), \tag{4.36}$$

$$(\boldsymbol{r})_j\sim-\frac{1}{\pi \mathrm{i}}\frac{1}{\varepsilon}(M-U_{j+1}+U_j+U_1), \tag{4.37}$$

and the jth elements of $\boldsymbol{c}_0=\int_{\mu_0}^{\infty^+}\boldsymbol{\omega}-\sum_{j=1}^{g}\int_{\mu_0}^{\mu_j}\boldsymbol{\omega}-\boldsymbol{K}$

$$c_{0j}\sim\sum_{i=1}^{g}\left[-\frac{1}{2\pi \mathrm{i}}\log\frac{\left(\lambda_0^+-\lambda_j\right)\left(\lambda_i^--\lambda_0\right)(\mu_i-\lambda_j)}{\left(\lambda_0^+-\lambda_0\right)\left(\lambda_i^--\lambda_j\right)(\mu_i-\lambda_0)}+\frac{1}{2}\right] \tag{4.38}$$

$$=\sum_{i=1}^{g}\left[-\frac{1}{2\pi \mathrm{i}}\log\frac{\left(\lambda_j-\lambda_0^+\right)\left(\lambda_i^--\lambda_0\right)(\mu_i-\lambda_j)}{\left(\lambda_0^+-\lambda_0\right)\left(\lambda_i^--\lambda_j\right)(\mu_i-\lambda_0)}\right], \tag{4.39}$$

where we have chosen the branch $\log(-1)=\pi\mathrm{i}$. Using the fact that μ_j $(j=1,2,\ldots,g)$ are the roots of $y_{N+1}(\lambda)=0$, we can immediately calculate the ultradiscrete limit of all the terms in

(4.39) except $(\mu_i - \lambda_j)$. (Note that $\mu_1 \rightarrowtriangle P_1 - P_0, \ldots, \mu_k \rightarrowtriangle P_k - P_{k-1}, \ldots, \mu_g \rightarrowtriangle -P_{g-1}$.) Indeed,

$$\prod_{i=1}^{g}\left(\lambda_i^- - \lambda_0\right) \rightarrowtriangle \sum_{i=1}^{g}(U_{i+1} - U_i) = -U_1,$$

$$\left|\prod_{i=1}^{g}\left(\lambda_i^- - \lambda_j\right)\right| \overset{\mathrm{u}}{\sim} \prod_{1\leqslant i<j}(\lambda_j - \lambda_i) \times (\lambda_j - \lambda_j^-) \times \prod_{j<i\leqslant g}(\lambda_i - \lambda_j)$$

$$= \prod_{1\leqslant i<j}(\lambda_j - \lambda_i) \times \frac{2m}{\prod_{l\neq j}|\lambda_l - \lambda_j|} \times \prod_{j<i\leqslant g}(\lambda_i - \lambda_j)$$

$$\rightarrowtriangle M - (U_1 - U_0)$$

and

$$\prod_{i=1}^{g}(\mu_i - \lambda_0) \overset{\mathrm{u}}{\sim} \prod_{i=1}^{g}\mu_i$$

$$\rightarrowtriangle \sum_{1\leqslant i\leqslant g}(P_i - P_{i-1})$$

$$= -P_0.$$

Unfortunately, it is not easy to calculate the terms $\prod_{i=1}^{g}(\mu_i - \lambda_j)$. For the time being, we treat these terms formally as

$$\prod_{i=1}^{g}(\mu_i - \lambda_j) \rightarrowtriangle \Xi_j.$$

We will prove a concrete expression for Ξ_j in the appendix. Thus, we obtain

$$c_{0j} \sim \frac{1}{2\pi\mathrm{i}}\frac{1}{\varepsilon}((g+1)M + gU_1 + U_0 - P_0 + g(U_j - U_{j+1}) - \Xi_j). \tag{4.40}$$

The fundamental decomposition of B is given as

$$-\frac{1}{\pi\mathrm{i}}\frac{1}{\varepsilon}\begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_g \end{pmatrix} = \Gamma_g \cdots \Gamma_3\Gamma_2 B \Gamma_2^t \Gamma_3^t \cdots \Gamma_g^t \tag{4.41}$$

where

$$A_k = \frac{k+1}{k}(-M - (k+1)U_k + kU_{k+1}), \qquad (k = 1, 2, \ldots, g, U_{g+1} = 0).$$

and

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1/2 & 1 & 0 & \ldots & 0 \\ -1/2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1/2 & 0 & 0 & \ldots & 1 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & -1/3 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1/3 & 0 & \ldots & 1 \end{pmatrix}, \ldots,$$

$$\Gamma_g = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1/g & 1 \end{pmatrix}.$$

Thus, using (4.36), we obtain

$$B^{-1}\boldsymbol{\nu} = \tfrac{1}{2}(\varsigma_1, \varsigma_2, \ldots, \varsigma_g)^t \tag{4.42}$$

where

$$\begin{aligned}
\varsigma_1 &= -\frac{1}{2}\frac{b_1}{c_1} + \frac{1}{2\cdot 3}\frac{b_2}{c_2} + \frac{1}{3\cdot 4}\frac{b_3}{c_3} + \cdots + \frac{1}{g(g+1)}\frac{b_g}{c_g},\\
\varsigma_2 &= -\frac{1}{3}\frac{b_2}{c_2} + \frac{1}{3\cdot 4}\frac{b_3}{c_3} + \cdots + \frac{1}{g(g+1)}\frac{b_g}{c_g},\\
\varsigma_3 &= -\frac{1}{4}\frac{b_3}{c_3} + \cdots + \frac{1}{g(g+1)}\frac{b_g}{c_g},\\
&\vdots\\
\varsigma_g &= -\frac{1}{g+1}\frac{b_g}{c_g},
\end{aligned}$$

with

$$b_k = U_0 - (k+1)U_k + kU_{k+1}, \qquad c_k = -M - (k+1)U_k + kU_{k+1}.$$

On the other hand, (4.32), (4.35) and (4.37) yield the important relation

$$\boldsymbol{r} = -\frac{1}{g+1}(\boldsymbol{b}_1 + \boldsymbol{b}_2 + \cdots + \boldsymbol{b}_g) \tag{4.43}$$

$$= -\frac{1}{N}(\boldsymbol{b}_1 + \boldsymbol{b}_2 + \cdots + \boldsymbol{b}_g), \tag{4.44}$$

where the $\boldsymbol{b}_j$ is jth column vector of the period matrix B.

4.2. Ultradiscretization of the theta function solution to the pd Toda

In this subsection, we will calculate the ultradiscrete limit of the meromorphic function of the form

$$\Psi_j(\boldsymbol{z}) := \frac{\partial}{\partial z_j}\log\frac{\theta(\boldsymbol{z}, B)}{\theta(\boldsymbol{z} - \frac{1}{N}(\boldsymbol{b}_1 + \cdots + \boldsymbol{b}_g), B)}, \qquad \text{where } \boldsymbol{z} = (z_1, \ldots, z_g)^t \tag{4.45}$$

rather than the theta function itself, because we want to ultradiscretize (2.20) with (4.44). We introduce the real matrix B° and the real vector $\boldsymbol{z}^\circ$ by $B = \mathrm{i}B^\circ$, $\boldsymbol{z} = \mathrm{i}B^\circ\boldsymbol{z}^\circ$. Starting from definition 2.2,

$$\theta(\boldsymbol{z}, B) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left(\pi \mathrm{i}\boldsymbol{n}^t B\boldsymbol{n} + 2\pi \mathrm{i}\boldsymbol{n}^t\boldsymbol{z}\right) \tag{4.46}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left(-\pi\boldsymbol{n}^t B^\circ\boldsymbol{n} - 2\pi\boldsymbol{n}^t B^\circ\boldsymbol{z}^\circ\right) \tag{4.47}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left(-\pi(\boldsymbol{n}+\boldsymbol{z}^\circ)^t B^\circ(\boldsymbol{n}+\boldsymbol{z}^\circ)\right)\exp\left(\pi\boldsymbol{z}^{\circ t}B^\circ\boldsymbol{z}^\circ\right), \tag{4.48}$$

and

$$\frac{\partial}{\partial z_j}\theta(\boldsymbol{z}, B) = 2\pi\mathrm{i}\sum_{\boldsymbol{n}\in\mathbb{Z}^g} n_j \exp(\pi\mathrm{i}\boldsymbol{n}^t B\boldsymbol{n} + 2\pi\mathrm{i}\boldsymbol{n}^t\boldsymbol{z}) \tag{4.49}$$

$$= 2\pi\mathrm{i} \sum_{\boldsymbol{n}\in\mathbb{Z}^g} n_j \exp(-\pi(\boldsymbol{n}+\boldsymbol{z}^\circ)^t B^\circ(\boldsymbol{n}+\boldsymbol{z}^\circ)) \exp(\pi \boldsymbol{z}^{\circ t} B^\circ \boldsymbol{z}^\circ). \tag{4.50}$$

Using these formulae, (4.45) becomes

$$\Psi_j(\boldsymbol{z}) = \frac{\theta_j(\boldsymbol{z}^\circ)\theta(\boldsymbol{z}^\circ-\boldsymbol{e}) - \theta(\boldsymbol{z}^\circ)\theta_j(\boldsymbol{z}^\circ-\boldsymbol{e})}{\theta(\boldsymbol{z}^\circ)\theta(\boldsymbol{z}^\circ-\boldsymbol{e})} \tag{4.51}$$

$$= 2\pi\mathrm{i}\frac{\sum_{\boldsymbol{n},\boldsymbol{m}}(n_j-m_j)\exp(-H(\boldsymbol{n}+\boldsymbol{z}^\circ))\exp(-H(\boldsymbol{m}+\boldsymbol{z}^\circ-\boldsymbol{e}))}{\sum_{\boldsymbol{n},\boldsymbol{m}}\exp(-H(\boldsymbol{n}+\boldsymbol{z}^\circ))\exp(-H(\boldsymbol{m}+\boldsymbol{z}^\circ-\boldsymbol{e}))} \tag{4.52}$$

where $\boldsymbol{e} = (1/N, 1/N, \ldots, 1/N)^t$, and $H(\boldsymbol{x}) = \boldsymbol{x}^t(\pi B^\circ)\boldsymbol{x}$.

Since $B \sim O(\varepsilon^{-1})$, we define $\Gamma(\boldsymbol{x}) := \lim_{\varepsilon\to 0^+} \varepsilon H(\boldsymbol{x})$. In (4.52), it turns out that the ultradiscrete behaviour of $\Psi_j(\boldsymbol{z})$ is strongly dependent on the term $(n_j - m_j)$. Recalling the fact that the period matrix must satisfy Im $B \geqslant 0$, we find that $\Gamma(\boldsymbol{x}) \geqslant 0, \forall \boldsymbol{x} \in \mathbb{R}^g$. Since Γ is a quadratic form over $\mathbb{R}^g$, we can order all the elements of $\mathbb{Z}^g \times \mathbb{Z}^g$ as

$$0 \leqslant \Gamma(\boldsymbol{z}^\circ+\boldsymbol{n}^{(1)}) + \Gamma(\boldsymbol{z}^\circ+\boldsymbol{m}^{(1)}-\boldsymbol{e}) \leqslant \Gamma(\boldsymbol{z}^\circ+\boldsymbol{n}^{(2)}) + \Gamma(\boldsymbol{z}^\circ+\boldsymbol{m}^{(2)}-\boldsymbol{e})$$
$$\leqslant \Gamma(\boldsymbol{z}^\circ+\boldsymbol{n}^{(3)}) + \Gamma(\boldsymbol{z}^\circ+\boldsymbol{m}^{(3)}-\boldsymbol{e}) \leqslant \ldots,$$

$((\boldsymbol{n}^{(i)}, \boldsymbol{m}^{(i)}) \in \mathbb{Z}^g \times \mathbb{Z}^g, i = 1, 2, \ldots)$.

Let $n_k^{(i)}$ and $m_k^{(j)}$ be the kth element of $\boldsymbol{n}^{(i)}$ and $\boldsymbol{m}^{(j)}$. Then, the asymptotic behaviour of $\Psi_j(\boldsymbol{z})$ is described as

$$\Psi_j(\boldsymbol{z}) \sim 2\pi\mathrm{i}\sum_{i=1}^{\infty}\left(n_j^{(i)} - m_j^{(i)}\right)\exp\left[\frac{-1}{\varepsilon}(\Gamma(\boldsymbol{n}^{(i)}+\boldsymbol{z}^\circ) + \Gamma(\boldsymbol{m}^{(i)}+\boldsymbol{z}^\circ-\boldsymbol{e}) - \Gamma(\boldsymbol{n}^{(1)}+\boldsymbol{z}^\circ) - \Gamma(\boldsymbol{m}^{(1)}+\boldsymbol{z}^\circ-\boldsymbol{e}))\right]. \tag{4.53}$$

Let

$$G_i(\boldsymbol{z}) := \Gamma(\boldsymbol{n}^{(i)}+\boldsymbol{z}^\circ) + \Gamma(\boldsymbol{m}^{(i)}+\boldsymbol{z}^\circ-\boldsymbol{e}) - \Gamma(\boldsymbol{n}^{(1)}+\boldsymbol{z}^\circ) - \Gamma(\boldsymbol{m}^{(1)}+\boldsymbol{z}^\circ-\boldsymbol{e}).$$

Since Γ is a positive definite quadratic form over $\mathbb{R}^g$, the set $\{\boldsymbol{x} \in \mathbb{R}^g | |\boldsymbol{x}| = R\} \subset \mathbb{R}^g$ is bounded for any $R > 0$. Thus, the set

$$\{i \in \mathbb{N} | \Gamma(\boldsymbol{n}^{(i)}+\boldsymbol{z}^\circ) + \Gamma(\boldsymbol{m}^{(i)}+\boldsymbol{z}^\circ) = R\}$$

is a finite set for any R. We arrange all the elements of $\{G_i(\boldsymbol{z})\}_{i\in\mathbb{N}}$ as

$$0 = G_1(\boldsymbol{z}) = \cdots = G_{\sigma(1)}(\boldsymbol{z}) < G_{\sigma(1)+1}(\boldsymbol{z}) = \cdots = G_{\sigma(2)}(\boldsymbol{z}) < \cdots.$$

Relation (4.53) becomes

$$\Psi_j(\boldsymbol{z}) \sim 2\pi\mathrm{i}\sum_{p=1}^{\infty}\sum_{l=\sigma(p-1)+1}^{\sigma(p)}\left(n_j^{(l)} - m_j^{(l)}\right)\exp\left[-\frac{G_l(\boldsymbol{z})}{\varepsilon}\right] \tag{4.54}$$

$$= 2\pi\mathrm{i}\sum_{p=1}^{\infty} I_p \exp\left[-\frac{G_{\sigma(p)}(\boldsymbol{z})}{\varepsilon}\right], \tag{4.55}$$

where $I_p = \sum_{l=\sigma(p-1)+1}^{\sigma(p)}\left(n_j^{(l)} - m_j^{(l)}\right)$.

Let

$$q(j) := \min\{p \in \mathbb{N} | I_p \neq 0\}.$$

To calculate the ultradiscrete limit of (2.20), we recall the calculations in section 4.1, and notice the following relation,

$$\int_{a_j} \lambda\omega_j \sim \lambda_j - \lambda_0,$$

which is obtained from $\omega_j \sim \omega_j^0 = \frac{1}{2\pi \mathrm{i}}\left\{\frac{1}{\lambda-\lambda_j} - \frac{1}{\lambda-\lambda_0}\right\} \mathrm{d}\lambda$. Hence, the coefficient $c_{j,g-1}$, defined in section 2, is found to be

$$c_{j,g-1} = \frac{\lambda_j - \lambda_0}{2\pi \mathrm{i}}.$$

Substituting these relations in (2.20), we obtain

$$I_{n+2}^t + V_{n+1}^t = \sum_{i=0}^{g}\left(I_i^0 + V_i^0\right) - \sum_{i=0}^{g}(\lambda_i - \lambda_0) - \sum_{j=1}^{g}\frac{\lambda_j - \lambda_0}{2\pi \mathrm{i}}\Psi_j(\boldsymbol{z}) \tag{4.56}$$

$$= \sum_{i=0}^{g}\lambda_i - \sum_{i=0}^{g}(\lambda_i - \lambda_0) - \sum_{j=1}^{g}\frac{\lambda_j - \lambda_0}{2\pi \mathrm{i}}\Psi_j(\boldsymbol{z}) \tag{4.57}$$

$$= (g+1)\lambda_0 - \sum_{j=1}^{g}\frac{\lambda_j - \lambda_0}{2\pi \mathrm{i}}\Psi_j(\boldsymbol{z}), \tag{4.58}$$

where $\boldsymbol{z} = n\boldsymbol{r} + t\boldsymbol{\nu} + \boldsymbol{c}(0)$.

The following formula gives an answer to the initial value problem of the pBBS.

Theorem 4.4. *The ultradiscretization of (4.58) is given by*

$$\min\left[Q_{n+2}^t, W_{n+1}^t\right] = \min_j\,[U_0 - U_1, (U_j - U_{j+1}) + \tilde{G}_{q(j)}(\boldsymbol{z})], \tag{4.59}$$

where $\tilde{G}_{q(j)}(\boldsymbol{z}) := \lim_{\varepsilon\to 0} G_{q(j)}(\boldsymbol{z})$ *with* $\boldsymbol{z} = n\boldsymbol{r} + t\boldsymbol{\nu} + \boldsymbol{c}(0)$.

Proof. By lemma 3.3, (4.59) is obvious when $U_0 - U_1$, and $(U_j - U_{j+1}) + \tilde{G}_{q(j)}(\boldsymbol{z})$, $(j = 1, 2, \ldots, g)$ are all distinct. In the general case, we have only to consider small perturbations as in remark 3.5. Since we can make $U_0 - U_1$ and $(U_j - U_{j+1}) + \tilde{G}_{q(j)}(\boldsymbol{z})$ all distinct by perturbing $U_j - U_{j+1}$ independently, we can conclude that (4.59) holds in the general case by the continuity of both sides of the equation. □

Remark 4.1. Note that $\Psi_j(n\boldsymbol{m} + t\boldsymbol{\nu} + \boldsymbol{c}(0))$ does not change under the translation

$$t\boldsymbol{\nu} \mapsto t\boldsymbol{\nu} + \boldsymbol{b}_k, \qquad \forall k$$

or equivalently

$$t(B^{-1})_k\boldsymbol{\nu} \mapsto t(B^{-1})_k\boldsymbol{\nu} + 1,$$

where $(B^{-1})_k$ is the kth column of B^{-1}.

5. Fundamental cycles of the periodic box-ball systems

5.1. Relative period

Let $\mathcal{Z}_N$ be a set of N-soliton states with no 0-soliton. We treat separately the set of N-soliton states with 0-solitons, which is denoted by $\mathcal{Z}_N{}^*$. A state $x \in \mathcal{Z}_N$ is expressed as

$$x = x_1x_2\cdots x_L \qquad \text{for} \quad x_i \in \{0, 1\}.$$

Using the translation map

$$\mathrm{S} : \mathcal{Z}_N \to \mathcal{Z}_N \qquad x_1x_2\cdots x_L \mapsto x_2\cdots x_Lx_1$$

which sends the first letter to the last, we define the set $\mathcal{T}(x) \subset \mathcal{Z}_N$ (for $x \in \mathcal{Z}_N$),

$$\mathcal{T}(x) := \{y \in \mathcal{Z}_N \mid \exists m \in \mathbb{Z}_{\geqslant 0} \text{ s.t. } \mathrm{S}^m(x) = y\}.$$

Let us denote the 10-elimination by $\mathrm{El} : \mathcal{Z}_N \to \mathcal{Z}_N \cup \bigcup_{n<N} \mathcal{Z}_n{}^*$, and the time evolution in the pBBS by $T : \mathcal{Z}_N \to \mathcal{Z}_N$. We also define $V : \mathcal{Z}_N \cup \mathcal{Z}_N{}^* \to \mathcal{Z}_N$ as the map which acts as the identity on $\mathcal{Z}_N$ and eliminates the 0-solitons in $\mathcal{Z}_N{}^*$.

Remark 5.1. El is bijective. $V \circ \mathrm{El}$ is surjective, but not injective.

Definition 5.1. *The fundamental cycle $f(x)$ of $x \in \mathcal{Z}_N$(or $\in \mathcal{Z}_N{}^*$) is the minimum positive integer p that satisfies $T^p(x) = x$. The relative period of $x \in \mathcal{Z}_N$(or $\in \mathcal{Z}_N{}^*$), $r(x)$, is the minimum positive integer q for which $T^q(x) \in \mathcal{T}(x)$.*

Remark 5.2. $r(x) \mid f(x)$ for any $x \in \mathcal{Z}_N$ because $x \in \mathcal{T}(x)$.

Remark 5.3. S and T commute. And S and El also commute. Consequently,

$$\mathcal{T}(x) = \mathcal{T}(y) \Leftrightarrow \mathcal{T}(\mathrm{El}(x)) = \mathcal{T}(\mathrm{El}(y)).$$

Since the method we presented in section 2 can only be used to calculate the relative period of pBBS systems, the following claims are important.

Lemma 5.1. *Let $x \in \mathcal{Z}_N$ be an N-soliton state of the pBBS. It holds that*

$$\mathcal{T}(\mathrm{El} \circ T^n(x)) = \mathcal{T}(T^n \circ \mathrm{El}(x)), \qquad n \in \mathbb{N}$$

or

$$\mathcal{T}(\mathrm{El}^{-1} \circ T^n(x)) = \mathcal{T}(T^n \circ \mathrm{El}^{-1}(x)).$$

Proof. Let Q_n^t and W_n^t ($n = 1, 2, \ldots, N, t \in \mathbb{N}$) be numbers defined by an N-soliton state x (see section 2.1, figure 2). Note that equations (2.1)–(2.4) give Q_n^{t+1} and W_n^{t+1} from Q_n^t and W_n^t. Obviously, when we replace Q_n^t and W_n^t to $Q_n^t - 1$ and $W_n^t - 1$, then these equations give $Q_n^{t+1} - 1$ and $W_n^{t+1} - 1$, which means

$$\mathcal{T}(\mathrm{El} \circ T(x)) = \mathcal{T}(T \circ \mathrm{El}(x)).$$

Using this formula, the former assertion is easily proved by induction. To prove the latter assertion, we start from the former assertion.

$$\begin{aligned}
\mathcal{T}(\mathrm{El} \circ T^n(x)) = \mathcal{T}(T^n \circ \mathrm{El}(x)) &\Leftrightarrow \exists m > 0 \text{s.t.} \mathrm{El} \circ T^n(x) = \mathrm{S}^m \circ T^n \circ \mathrm{El}(x) \\
&\Leftrightarrow \mathrm{El} \circ T^n \circ \mathrm{El}^{-1}(y) = \mathrm{S}^m \circ T^n(y) \\
&\Leftrightarrow T^n \circ \mathrm{El}^{-1}(y) = \mathrm{El}^{-1} \circ \mathrm{S}^m \circ T^n(y) \qquad (5.1)
\end{aligned}$$

where $y = \mathrm{El}(x)$. Recalling remark 5.3, (5.1) is equivalent to $T^n \circ \mathrm{El}^{-1}(y) = \mathrm{S}^m \circ \mathrm{El}^{-1} \circ T^n(y)$, which completes the proof. □

Proposition 5.2. *If a state $x \in \mathcal{Z}_N$ satisfies the condition $\mathrm{El}(x) \in \mathcal{Z}_{N-1}{}^*$, then $r(x) = f(\mathrm{El}(x))$.*

Proof. Let $\tilde{x} = \mathrm{El}(x)$. Since $\mathrm{El}(x) \in \mathcal{Z}^*_{N-1}$, $\tilde{x}$ has exactly one 0-soliton. If $T^p(\tilde{x}) = \tilde{x}$,

$$\mathcal{T}(x) = \mathcal{T}(\mathrm{El}^{-1}(\tilde{x})) = \mathcal{T}(\mathrm{El}^{-1} \circ T^p(\tilde{x})) = \mathcal{T}(T^p \circ \mathrm{El}^{-1}(\tilde{x})) = \mathcal{T}(T^p(x)).$$

So, $r(x) \leqslant f(\tilde{x})$. Conversely, if $\mathcal{T}(x) = \mathcal{T}(T^q(x))$, it follows that

$$\mathcal{T}(\tilde{x}) = \mathcal{T}(\mathrm{El}(x)) = \mathcal{T}(\mathrm{El} \circ T^q(x)) = \mathcal{T}(T^q \circ \mathrm{El}(x)) = \mathcal{T}(T^q(\tilde{x})).$$

Since a 0-soliton does not move under the time evolution, the fact that $\tilde{x}$ has exactly one 0-soliton leads to $\tilde{x} = T^q(\tilde{x})$. So, $r(x) \geqslant f(\tilde{x})$. □

Remark 5.4. In the proof of proposition 5.2, we conclude that $\mathcal{T}(\tilde{x}) = \mathcal{T}(T^q(\tilde{x})) \Rightarrow \tilde{x} = T^q(\tilde{x})$. This claim fails in the case where there are more than two 0-solitons in the sequence of $\mathrm{El}(x)$, where are arranged symmetrically. In this situation,

$$r(x) \lneqq f(\tilde{x}).$$

We call this symmetry 'internal symmetry'. Internal symmetry makes the problem more complicated. We do not consider this symmetry in the present paper.

The statement of proposition 5.2 can be generalized as follows.

Corollary 5.3. *If a state $x \in \mathcal{Z}_N$ satisfies the condition $\mathrm{El}(x) \in \mathcal{Z}_n^*$, where $n < N$, and x is without internal symmetry, then $r(x) = f(\mathrm{El}(x))$.*

By virtue of corollary 5.3, the fundamental period of the pBBS can be obtained from the relative period of the corresponding pBBS.

5.2. Formula for the fundamental period

Recall the definition of the Young diagram associated with the state of the pBBS (section 2.3). We also define

$$n_l := \{\text{the number of the rows of which length is } L_l\},$$

$$l_0 := L - \sum_{j=1}^{l} 2p_j, \qquad l_j := L_j - L_{j+1}, \qquad N_j := l_0 + \sum_{l=1}^{j} 2n_l(L_l - L_{j+1}).$$

The fundamental cycle of $x \in \mathcal{Z}_N$ can be described by using the data of the corresponding Young diagram. In fact, the following formula gives the fundamental cycle of the pBBS system.

Theorem 5.4. *Let $x \in \mathcal{Z}_N$ be a N-soliton pBBS without internal symmetry. Then*

$$f(x) = \mathrm{LCM}\left(\frac{N_s N_{s-1}}{l_s l_0}, \frac{N_{s-1} N_{s-2}}{l_{s-1} l_0}, \ldots, \frac{N_1 N_0}{l_1 l_0}, 1\right).$$

Remark 5.5. Recalling proposition 5.2, the formula in proposition 5.4 is equivalent to

$$r(x) = \mathrm{LCM}\left(\frac{N_{s-1} N_{s-2}}{l_{s-1} l_0}, \ldots, \frac{N_1 N_0}{l_1 l_0}, 1\right)$$

because one can obtain the Young diagram corresponding to $\mathrm{V} \circ \mathrm{El}(x)$ by eliminating the first column of the Young diagram corresponding to x.

Though this formula was first obtained by elementary combinatorial methods [5], we can obtain the same formula using a different method relying on the results of the previous sections.

From (2.20), (4.42) and remark 4.1, the relative period $r(x)$ satisfies

$$r(x) = \mathrm{LCM}\left(\frac{2}{\varsigma_1}, \frac{2}{\varsigma_2}, \ldots, \frac{2}{\varsigma_g}, 1\right), \tag{5.2}$$

where ς_j are numbers defined by (4.42). We use the following lemmas to prove theorem 5.4.

Lemma 5.5. $\varsigma_k = \varsigma_{k+1} \Leftrightarrow$ *the lengths of kth and* $(k+1)$*th rows from the bottom in the Young diagram are equal.*

Proof.

$$\begin{aligned}\varsigma_k = \varsigma_{k+1} &\Leftrightarrow -\frac{1}{k+1}\frac{b_k}{c_k} + \frac{1}{(k+1)(k+2)}\frac{b_{k+1}}{c_{k+1}} = -\frac{1}{k+2}\frac{b_{k+1}}{c_{k+1}}\\ &\Leftrightarrow \frac{b_k}{c_k} = \frac{b_{k+1}}{c_{k+1}}\\ &\Leftrightarrow \frac{U_0 - (k+1)U_k + kU_{k+1}}{-M - (k+1)U_k + kU_{k+1}} = \frac{U_0 - (k+2)U_{k+1} + (k+1)U_{k+2}}{-M - (k+2)U_{k+1} + (k+1)U_{k+2}}\\ &\Leftrightarrow U_k - U_{k+1} = U_{k+1} - U_{k+2}.\end{aligned}$$

Lemma 3.12 completes the proof. □

The following lemma is almost trivial. We omit the proof.

Lemma 5.6. *Let* p, p', p'', q, q', q'' *be integers, and* p, p', p'' *are relatively prime to* q, q', q''*, respectively. If*

$$\frac{q}{p} - \frac{q'}{p'} = \frac{q''}{p''},$$

then $\mathrm{LCM}(q, q') = \mathrm{LCM}(q, q'')$.

Proof of theorem 5.4. First, we prove the theorem for the case where

$$i \neq j \Rightarrow \varsigma_i \neq \varsigma_j. \tag{5.3}$$

Starting from (5.2),

$$r(x) = \mathrm{LCM}\left(\frac{2}{\varsigma_1}, \ldots, \frac{2}{\varsigma_g}, 1\right) \tag{5.4}$$

$$= \mathrm{LCM}\left(\frac{2}{\varsigma_1 + \sum_{j=1}^{g}\varsigma_j}, \frac{2}{\varsigma_1 - \varsigma_2}, \ldots, \frac{2}{\varsigma_k - \varsigma_{k+1}}, \ldots, \frac{2}{\varsigma_{g-1} - \varsigma_g}, 1\right) \tag{5.5}$$

by lemma 5.6. From (4.42),

$$\begin{aligned}\varsigma_k - \varsigma_{k+1} &= -\frac{1}{k+1}\frac{b_k}{c_k} + \left(\frac{1}{(k+1)(k+2)} + \frac{1}{k+2}\right)\frac{b_{k+1}}{c_{k+1}}\\ &= \cdots\\ &= \frac{(-M - U_0)(U_{k+2} - 2U_{k+1} + U_k)}{(-M - (k+1)U_k + kU_{k+1})(-M - (k+2)U_{k+1} + (k+1)U_{k+2})}.\end{aligned}$$

On the other hand, by definition of l_j, N_j and lemma 3.12, we obtain

$$l_j = L_{j+1} - L_j = U_j - U_{j+1} - (U_{j-1} - U_j) = -U_{j+1} + 2U_j - U_{j-1},$$

$$l_0 = L - 2U_0,$$

and

$$N_j = l_0 + 2\sum_{l=1}^{j} n_l(L_l - L_{j+1})$$
$$= L - 2(j+1)U_j + 2jU_{j+1} \quad (\because (5.3) \overset{\text{Lem.5.5}}{\Longrightarrow} n_j = 1).$$

Recalling $M = -L/2$, we obtain

$$\varsigma_j - \varsigma_{j+1} = \frac{2l_0 l_{j+1}}{N_{j+1}N_j}, \qquad (j = 1, 2, \ldots, g-1).$$

And, by definition of ς_j ((4.42)), we derive

$$\varsigma_1 + \sum_{j=1}^{g} \varsigma_j = -\frac{b_1}{c_1} = -\frac{U_0 - 2U_1 + U_2}{-M - 2U_1 + U_2} = \frac{2l_1}{N_1}\left(= \frac{2l_1 l_0}{N_1 N_0}\right).$$

By (5.5), it follows that

$$r(x) = \mathrm{LCM}\left(\frac{N_g N_{g-1}}{l_g l_0}, \ldots, \frac{N_1 N_0}{l_1 l_0}, 1\right).$$

The fact that (5.3) $\Rightarrow s = N = g+1$ completes the proof under condition (5.3).

For general cases, let us define $\varrho(i)$ $(i = 1, 2, \ldots, s)$ by

$$\varsigma_1 = \cdots = \varsigma_{\varrho(1)} > \varsigma_{\varrho(1)+1} = \cdots = \varsigma_{\varrho(2)} > \varsigma_{\varrho(2)+1} = \cdots > \varsigma_{\varrho(s-1)-1} = \cdots = \varsigma_{\varrho(s)},$$

and $\varrho(0) := 0$. We obtain

$$l_j = -U_{\varrho(j+1)} + 2U_{\varrho(j)} - U_{\varrho(j-1)},$$

and

$$N_j = l_0 + 2\sum_{l=1}^{j} n_l(L_l - L_{\varrho(j+1)})$$
$$= l_0 + 2\sum_{k=1}^{s} \sum_{\varrho(k-1)+1 \leqslant l \leqslant \varrho(k)} (\varrho(k) - \varrho(k-1))(L_l - L_{\varrho(j+1)})$$
$$= L - 2(\varrho(j)+1)U_{\varrho(j)} + 2\varrho(j)U_{\varrho(j)+1}.$$

We can complete the proof in a similar manner to that of the previous case. □

Acknowledgments

The authors are very grateful to Professor Ralph Willox for helpful comments on this paper.

Appendix. The proofs of the remained lemmas

A.1. Proof of lemma 3.12

To prove lemma 3.12, we investigate the 10-elimination (section 2.3) in detail. Let us consider a state consisting of N blocks of consecutive 1's and 0's, which are arranged alternatingly. We denote the length of the kth block of consecutive 1's by Q_k and kth consecutive 0's by W_k. In order to show how these blocks are reconstructed by 10-eliminations, it is convenient to draw a graph which consists of nodes and links. For example, let us consider a state where $N = 3$, $Q_1 = 5$, $W_1 = 3$, $Q_2 = 1$, $W_2 = 2$, $Q_3 = 6$ and $W_3 = 12$ (see figure A1). The blocks

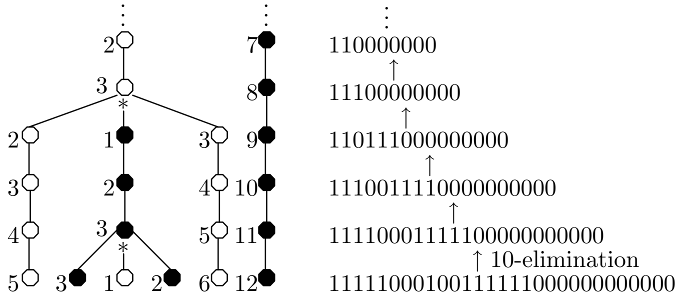

Figure A1. A state of the pBBS with $N = 3$ and the associated graph. The nodes at the bottom the graph are associated with $(Q_1, W_1, Q_2, W_2, Q_3, W_3) = (5, 3, 1, 2, 6, 12)$.

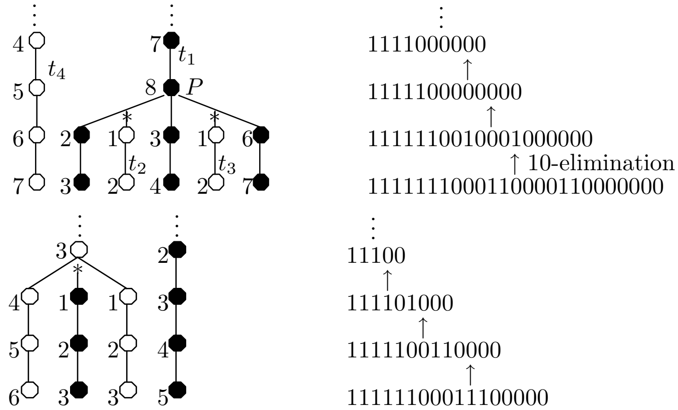

Figure A2. Examples of the graph associated with two typical states. In the first case, two blocks of 0's disappear simultaneously. In the second case, two adjacent blocks disappear simultaneously, which would require writing the $*$ twice. We shall write it only once.

of 1's are represented by white nodes, and those of 0's by black nodes. A number is associated with each node. The numbers in the bottom of the graph are equal to $Q_1, W_1, \ldots, Q_N, W_N$, respectively. Going up we arrange by one step, these numbers decrease by 1. The sign '$*$' means zero, and a blocks of consecutive 1's or 0's disappears at the point where $*$ appears. When one block disappears, the two blocks adjacent to a '$*$' join together. Figure A2 show examples of the graphs associated with a typical state.

Let us define several terms relating to this associated graph.

Definition A.1. *A tree is a connected component in the associated graph.*

Note that any tree has exactly one $*$.

Remark A.1. Only two types of tree can exist. One is a tree consisting of white nodes, and the other is a tree of black nodes. We denote the 'white tree' as a'w-tree', and 'black tree' as a 'b-tree'.

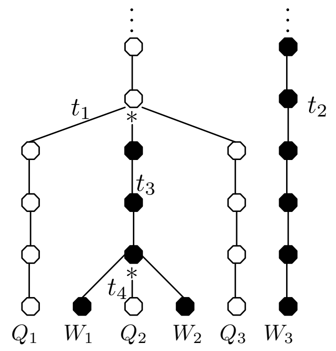

Figure A3. An example clarifying definition A.2.

Definition A.2. *Let P be a node, and t be a tree in the associated graph. The height of P, denoted by* $\mathrm{Ht}(P)$*, is the number of links in the path from P to the bottom of the graph.*

And the height of t, denoted by $\mathrm{Ht}(t)$*, is the height of $*$ contained within t.*

Let us denote by Φ_x the graph associated with the state $x \in \mathcal{Z}_N$. We introduce a semiordering on the set of trees by

$$t, t' \in \Phi_x, \qquad t \succ t' \Leftrightarrow t \quad \text{straddles} \quad t'.$$

Now we define two important sets.

Definition A.3. *Let $t \in \Phi_x$ be a tree, then we define a set of trees as*

$$\mathrm{Und}(t) := \{s \in \Phi_x : \textit{tree} \mid s \preceq t\},$$

and a set of integers as

$$\mathrm{Ft}(t) := \left\{ A_{\sigma(i)} \in X \,\middle|\, \begin{array}{c} A_{\sigma(i)} \textit{ is an associated number with} \\ \textit{a node at the foot of tree } t. \end{array} \right\},$$

with $X = \{A_{\sigma(i)}\}_{i=1}^{2N}$, $A_{2l-1} = Q_l$, and $A_{2l} = W_l$.

Example. In figure A3, t_1 and t_4 are white trees, and t_2 and t_3 are black trees. The trees have the relation $t_4 \prec t_3 \prec t_1$, $\mathrm{Und}(t_1) = \{t_3, t_4\}$, $\mathrm{Ft}(t_1) = \{Q_1, Q_3\}$, etc. The height of t_3 and t_4 are given as $\mathrm{Ht}(t_3) = 4$, and $\mathrm{Ht}(t_4) = 1$. Note that

$$t \quad \text{is a w-tree} \qquad \Leftrightarrow \qquad \mathrm{Ft}(t) \subset \{Q_i\}_{i=1}^N, \tag{A.1}$$

and

$$t \quad \text{is a b-tree} \qquad \Leftrightarrow \qquad \mathrm{Ft}(t) \subset \{W_i\}_{i=1}^N. \tag{A.2}$$

Lemma A.1. *Let $t \in \Phi_x$ be a tree. The number of links in t is equal to $\sum_{A_{\sigma(i)} \in \mathrm{Ft}(t)} A_{\sigma(i)}$.*

Proof. This is a natural consequence of the definition of the associated graph. □

Remember that a maximal element ξ in a semiorderd set X is the element which satisfies

$$\xi' \in X, \qquad \xi \preceq \xi' \Rightarrow \xi = \xi'.$$

Remark A.2. Let $t \in \Phi_x$ be a w-tree, and $s \in \Phi_x$ be a b-tree. Then any maximal element of $\mathrm{Und}(t)\backslash\{t\}$ is a b-tree, and any maximal element of $\mathrm{Und}(s)\backslash\{t\}$ is a w-tree.

We call the node which is connected with more than 2 links the branch point. We define the multiplicity of the branch point P as

$$m_P := \{\text{the number of links connected to } P\} - 2.$$

Note that

$$\{\text{the number of maximal element in } \mathrm{Und}(t)\backslash\{t\}\} = \sum_{P:\,\text{branch pt. in } t} m_P.$$

Example. In figure A2, the multiplicity m_P is equal to 2. The maximal elements of $\mathrm{Und}(t)\backslash\{t\}$ are t_2 and t_3. Note that $\mathrm{Ht}(P) = \mathrm{Ht}(t_2) = \mathrm{Ht}(t_3) = 2$.

Let us define

$$\mathrm{Ft}(\mathrm{Und}(t)) := \bigcup_{s \in \mathrm{Und}(t)} \mathrm{Ft}(s),$$

where $t \in \Phi_x$ is a tree.

Lemma A.2. *Let $t \in \Phi_x$ be a w-tree, and $s \in \Phi_x$ be a b-tree. Then,*

$$\sum_{t' \in \mathrm{Und}(t)} \mathrm{Ht}(t') = \sum_{Q_i \in \mathrm{Ft}(\mathrm{Und}(t)) \cap \{Q_j\}} Q_i, \tag{A.3}$$

and

$$\sum_{s' \in \mathrm{Und}(s)} \mathrm{Ht}(s') = \sum_{W_i \in \mathrm{Ft}(\mathrm{Und}(s)) \cap \{W_j\}} W_i. \tag{A.4}$$

Proof. Let $h := \#\mathrm{Und}(t)$. We prove the lemma by induction of h.

When $h = 1$, a w-tree t is a straight segment of length Q_j, where $\{Q_j\} = \mathrm{Und}(t)$. Then, $\mathrm{Ht}(t) = Q_j$. Hence, recalling (A.1), we can conclude that (A.3) is true immediately. Similarly, (A.4) is true for a b-tree s.

Suppose that (A.3) and (A.4) hold for $h = 1, 2, \ldots, p$. Let t be a w-tree with $\#\mathrm{Und}(t) = p + 1$. Denote the branch points which belong to t by $P_1, P_2, \ldots, P_r$.

From lemma A.1, we obtain

$$\{\text{the number of links consisted in } t\} = \sum_{Q_i \in \mathrm{Ft}(t)} Q_i$$
$$= \mathrm{Ht}(t) + \sum_{j=1}^{r} \mathrm{Ht}(P_j). \tag{A.5}$$

(See figure A4.) Since we can assume that the maximal elements $s_1, s_2, \ldots, s_r \in \mathrm{Und}(t)\backslash\{t\}$ satisfy $\mathrm{Ht}(s_j) = \mathrm{Ht}(P_j)$ $(j = 1, 2, \ldots, r)$, (A.5) becomes

$$\sum_{Q_i \in \mathrm{Ft}(t)} Q_i = \mathrm{Ht}(t) + \sum_{j=1}^{r} \mathrm{Ht}(s_j). \tag{A.6}$$

On the other hand, we easily find that

$$\sum_{j=1}^{r} \mathrm{Ht}(s_j) = \sum_{t' \in \mathrm{Und}(t)\backslash\{t\}} \mathrm{Ht}(t') - \sum_{j=1}^{r} \left(\sum_{t'' \in \mathrm{Und}(s_j)\backslash\{s_j\}} \mathrm{Ht}(t'') \right). \tag{A.7}$$

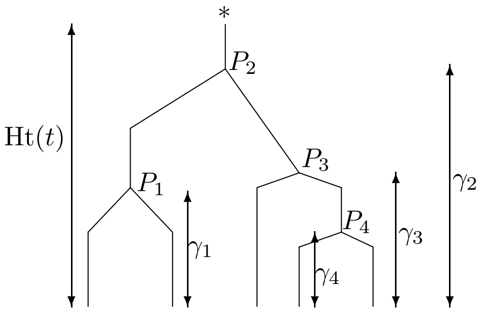

Figure A4. An example of tree t. ($\gamma_j := \mathrm{Ht}(P_j)$).

Let $\tilde{t}_{jk}$ ($k = 1, 2, \ldots, l_j$) be maximal elements of $\mathrm{Und}(s_j)\backslash\{s_j\}$. By the induction hypothesis and (A.3) we obtain

$$\sum_{j=1}^{r}\left(\sum_{t''\in\mathrm{Und}(s_j)\backslash\{s_j\}}\mathrm{Ht}(t'')\right) = \sum_{j=1}^{r}\sum_{k=1}^{l_k}\sum_{t''\in\mathrm{Und}(\tilde{t}_{jk})}\mathrm{Ht}(t'') \tag{A.8}$$

$$= \sum_{j=1}^{r}\sum_{k=1}^{l_k}\sum_{Q_i\in\mathrm{Ft}(\mathrm{Und}(\tilde{t}_{jk}))\cap\{Q_n\}} Q_i \tag{A.9}$$

$$= \sum_{Q_i\in(\mathrm{Ft}(\mathrm{Und}(t)\backslash\{t\}))} Q_i. \tag{A.10}$$

Substituting (A.7) and (A.10) to (A.6), we find that (A.3) holds for $h = p+1$. Equation (A.4) can be proved in a similar manner. □

Let

$$H_k := \{\text{the height of the } k\text{th smallest tree in } \Phi_x\}.$$

Note that the proof of lemma 3.12 is completed by proving the following two formulae (A.11) and (A.12):

$$\min\left\{\sum_{\{A_{\sigma(i)}\}\in\mathcal{B}(N-k,N)} A_{\sigma(i)}\right\} = H_1 + \cdots + H_k, \tag{A.11}$$

$$H_1 + \cdots + H_k = \left\{\begin{array}{l}\text{the number of boxes}\\ \text{from under to the } k\text{th step in the Young diagram}\end{array}\right\}. \tag{A.12}$$

Proof of (A.11). Let $\{A_{\sigma(i)}\}^*$ be an element of $\mathcal{B}(N-k, N)$ which satisfies

$$\min\left\{\sum_{\{A_{\sigma(i)}\}\in\mathcal{B}(N-k,N)} A_{\sigma(i)}\right\} = \sum_{\{A_{\sigma(i)}\}^*\in\mathcal{B}(N-k,N)} A_{\sigma(i)}.$$

Without loss of generality, we can assume $Q_\alpha \in \{A_{\sigma(i)}\}^*$ for some α. Let $t \in \Phi_x$ be a tree with $Q_\alpha \in \mathrm{Ft}(t)$. It follows that

$$Q_\beta \in \mathrm{Ft}(\mathrm{Und}(t)) \Rightarrow Q_\beta \in \{A_{\sigma(i)}\}^*. \tag{A.13}$$

($\because$ Otherwise, there exists $Q_{\beta_1}, \ldots, Q_{\beta_l}$ with $Q_{\beta_j} \in \mathrm{Ft}(\mathrm{Und}(t))$ and $Q_{\beta_j} \notin \{A_{\sigma(i)}\}^*$. Let P_j be the nearest branch point to Q_{β_j}. Without loss of generality, we can assume $\mathrm{Ht}(P_1) = \min(\mathrm{Ht}(P_j))$. Let us define the subgraph

$$\Phi_x^* := \left\{ P : \text{node} \;\middle|\; \begin{array}{l} \text{the path } P_1 \to P \text{ consists of links} \\ \text{extending below } P_1 \end{array} \right\},$$

and

$$\Phi_x^{**} := \{\text{the node and links straddled by } \Phi_x^*.\}.$$

Note that $\Phi_x^{**} \backslash \{Q_{\beta_1}\} \subset \{A_{\sigma(i)}\}^*$. The inequality

$$\sum_{Q_j \in \Phi_x^{**} \backslash \{Q_{\beta_1}\}} Q_j > \sum_{W_i \in \Phi_x^{**}} W_i$$

leads to a contradiction with the definition of $\{A_{\sigma(i)}\}^*$.)

Relation (A.13) claims

$$\{A_{\sigma(i)}\}^* = \left\{ \coprod_j (\mathrm{Ft}(\mathrm{Und}(t_j)) \cap \{Q_n\}) \right\} \amalg \left\{ \coprod_j (\mathrm{Ft}(\mathrm{Und}(s_m)) \cap \{W_n\}) \right\}$$

for some w-trees t_j and b-trees s_m. From lemma A.2, we obtain

$$\sum_{\{A_{\sigma(i)}\}^* \in \mathcal{B}(N-k,N)} A_{\sigma(i)} = H_{\tau_1} + H_{\tau_2} + \cdots + H_{\tau_l},$$

for some τ. It completes the proof of (A.11). □

Proof of (A.12). Note that the number of boxes below the kth step in the Young diagram is equal to the number of bridges connected to the blocks which disappear by 10-eliminations, when the number of solitons becomes $N-k$ (see figure 5). By definition of the associated graph Φ_x, it is clear that equation (A.12) holds. □

From proposition 3.9, (A.11) and (A.12), lemma 3.12 is proved.

A.2. Proof of proposition 3.7 and 3.10

In this subsection, we give the proof of proposition 3.7 and 3.10. Recall (2.15):

$$\begin{cases} x_{n+1} = (\lambda - a_n)x_n - b_n x_{n-1} \\ y_{n+1} = (\lambda - a_n)y_n - b_n y_{n-1} \end{cases}$$

and the initial condition

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $X_j = \lambda - a_j$ and $Y_j = -b_j$. For convenience, we denote X_j by (j), and Y_j by $(j-1, j)$ in this section. For example, $(a)(b-1, b)$ means $X_a \cdot Y_b$.

Let

$$\Omega_{jk} := \left\{ \begin{array}{r} \{i_1, i_2, \ldots, i_s, j_1, j_2, \ldots, j_t\} \\ \subset \mathbb{Z}/N\mathbb{Z} \end{array} \;\middle|\; \begin{array}{r} 2s + t = k + 1 - j, \\ \{i_1 - 1, i_1, \ldots, i_s - 1, i_s, j_1, \ldots, j_t\} \\ = \{j, j+1, \ldots, j_k\} \end{array} \right\}.$$

We define

$$((j, j+1, \ldots, k)) := \sum_{\{i_1, i_2, \ldots, i_s, j_1, \ldots, j_t\} \in \Omega_{jk}} (i_1 - 1, i_1)(i_2 - 1, i_2) \cdots (i_s - 1, i_s)(j_1)(j_2) \cdots (j_t).$$

For example, $((1,2,3)) = (1)(2)(3) + (1,2)(3) + (1)(2,3) = X_1X_2X_3 + Y_2X_3 + X_1Y_3$. We also defined $((2,3,\ldots,k))$ by a similar rule. (For example, $((2,3)) = (2)(3) + (2,3) = X_2X_3 + Y_3$.)

Lemma 14. *If* $2 \leqslant n \leqslant N+1$*, then* $x_n = ((1,2,\ldots,n-1))$*. And if* $3 \leqslant n \leqslant N+1$*, then* $y_n = Y_1 \times ((2,3,\ldots,n-1))$.

Proof. We prove the lemma by induction. When $n = 2$, $x_2 = (\lambda - a_1)x_1 - b_1x_0 = \lambda - a_1 = X_1 = ((1))$. Since $x_{n+1} = X_nx_n + Y_nx_{n-1}$, from the induction hypothesis, $x_{n+1} = (n)\cdot((1,2,\ldots,n-1)) + (n-1,n)\cdot((1,2,\ldots,n-2)) = ((1,2,\ldots,n))$. Hence $x_n = ((1,2,\ldots,n))$ holds for any n. We can prove the assertion for y_n in a similar fashion. □

If $n > N$, n should be considered as an element of $\mathbb{Z}/N\mathbb{Z}$ because of the periodic boundary condition $a_{N+j} = a_j, b_{N+j} = b_j$. In this sense, we may write $Y_1 = (N,1)$. Propositions 3.7 and 3.10 are proved immediately from lemma 14. In fact, the relation $y_{N+1} = Y_1 \times ((2,3,\ldots,N))$ is equivalent to proposition 3.10. In order to prove proposition 3.7, note that $\Delta(\lambda) = x_{N+1} + y_N = ((1,2,\ldots,N)) + (N,1)((2,3,\ldots,N-1))$. Let us decompose $\mathcal{A}(N) \subset 2^{\mathbb{Z}/N\mathbb{Z}}$ into two disjoint sets

$$\mathcal{A}(N) = \mathcal{A}' \sqcup \mathcal{A}''$$

where $\mathcal{A}'$ consists of the subsets ($\subset \mathbb{Z}/N\mathbb{Z}$) that include $(N,1)$, and $\mathcal{A}'' := \mathcal{A} - \mathcal{A}'$. Proposition 3.7 is obtained by the fact that

$$\sum_{(j_1-1,j_1,\ldots,j_k-1,j_k)\in\mathcal{A}'} Y_{j_1}\ldots Y_{j_k}X_{i_1}\ldots X_{i_{N-2k}} = (N,1)((2,3,\ldots,N-1))$$

and

$$\sum_{(j_1-1,j_1,\ldots,j_k-1,j_k)\in\mathcal{A}''} Y_{j_1}\ldots Y_{j_k}X_{i_1}\ldots X_{i_{N-2k}} = ((1,2,\ldots,N)).$$

A.3. Calculation of Ξ_j

Recall that a_j $(j = 0,1,\ldots,g)$ are the roots of $\Delta(\lambda) = 0$, and μ_j $(j = 1,2,\ldots,g)$ are the roots of $y_{N+1}(\lambda) = 0$. Hence,

$$\prod_{i=1}^{g}(\mu_j - \lambda_i) = \frac{\Delta(\mu_j)}{\mu_j - \lambda_0} = \frac{[((1,2,\ldots,N)) + (N,1)((2,3,\ldots,N-1))]_{\lambda=\mu_j}}{\mu_j - \lambda_0},$$

where $N = g+1$.

Since

$$((2,3,\ldots,N))|_{\lambda=\mu_j} = 0$$

and

$$((1,2,\ldots,N)) = (1)((2,3,\ldots,N)) + (1,2)((3,4,\ldots,N)),$$

we find

$$\prod_{i=1}^{g}(\mu_j - \lambda_i) = \frac{[(1,2)((3,4,\ldots,N)) + (N,1)((2,3,\ldots,N-1))]_{\lambda=\mu_j}}{\mu_j - \lambda_0}.$$

Note that $(j) = \lambda - (I_{j+1} + V_j)$ and $(j-1,j) = -I_jV_j$.

We show some examples here.

Example. If $N = 2(g = 1)$,

$$\begin{aligned}-(\mu_1 - \lambda_1) &= \frac{I_0V_0 + I_1V_1}{\mu_1 - \lambda_0} \overset{\text{u}}{\sim} \frac{I_0V_0 + I_1V_1}{\mu_1} \\ &\rightarrow -\min\left\{Q_0^0 + W_0^0, Q_1^0 + W_1^0\right\} - (-P_1) \\ &\equiv \Xi_1.\end{aligned}$$

If $N = 3$ $(g = 2)$,

$$\prod_{i=1}^{2}(\mu_j - \lambda_i) = \frac{-I_2V_2(\mu_j - (I_1 + V_0)) - I_1V_1(\mu_j - (I_0 + V_2))}{\mu_j - \lambda_0}. \tag{A.14}$$

To calculate the ultradiscrete limit of the right-hand side of (A.14), we have to compare the magnitude of each term.

For example, when $j = 1$, $(P_2 - P_1) < \min\left\{Q_1^0, W_0^0\right\}$, and $Q_2^0 + W_2^0 + P_2 - P_1 < Q_1^0 + W_1^0 + \min\left\{P_2 - P_1, Q_0^0, W_2^0\right\}$, then

$$\Xi_1 = Q_2^0 + W_2^0 + P_2 - P_1.$$

References

[1] Kimijima T and Tokihiro T 2002 Initial-value problem of the discrete periodic Toda equation and its ultradiscretization *Inverse Problems* **18** 1705–32

[2] Kasman A and Lafortune S 2006 When is negativity not a problem for the ultradiscrete limit? *J. Math. Phys.* **47** 103510

[3] Mumford D 1983 *Tata lectures on Theta Functions* vol 1 (Boston, MA: Birkhäuser)

[4] Fay J D 1973 *Theta-Functions on Riemann Surfaces (Lecture Notes of Mathmatics* vol 352*)* (Berlin: Springer)

[5] Yoshihara D, Yura F and Tokihiro T 2003 Fundamental cycle of a periodic box-ball system *J. Phys. A: Math. Gen.* **36** 99–121

[6] Toda M 1978 *Hisenkei kôshirikigaku* (Tokyo: Iwanami) (in Japanese)